Option Pricing in Stochastic Volatility Models Driven by Fractional Jump-Diffusion Processes

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Abstract - **In this paper, we propose a fractional stochastic volatility jump-diffusion model which extends the Bates (1996) model, where we model the volatility as a fractional process. Extensive empirical studies show that the distributions of the logarithmic returns of financial asset usually exhibit properties of self-similarity and long-range dependence and since the fractional Brownian motion has these two important properties, it has the ability to capture the behavior of underlying asset price. Further incorporating jumps into the stochastic volatility framework gives further freedom to financial mathematicians to fit both the short and long end of the implied volatility surface. We propose a stochastic model which contains both fractional and jump process. Then we price options using Monte Carlo simulations along with a variance reduction technique (antithetic variates). We use market data from the S&P 500 index and we compare our results with the Heston and Bates model using error measures. The results show our model greatly outperforms previous models in terms of estimation accuracy.**

Keywords - Hurst exponent, Jump-Diffusion, Fractional stochastic volatility model, Option pricing, Long-range dependence.

1. Introduction

As the market for financial derivatives continues to grow, the success of option pricing models at estimating the value of option premiums is under examination. Since Black and Scholes (Black and Scholes, 1973) published their seminal article on option pricing based on Brownian motion in 1973, there has been an explosion of theoretical and empirical work on option pricing. However, In the Black and Scholes model, a basic assumption is that the volatility is constant. It was soon discovered that this assumption does not allow matching an entire option chain (option values for different strike values) (See: Heston, 1993; Bates, 1993). i.e. the well-known volatility smile/skew phenomena. This phenomena has been considered as a result of the non-Gaussian behavior of the distribution of the return rate: compared to the perfect bell-curve of normal distribution, the ground true distribution (although not observable) is skewed and leptokurtic with long term memory and correlated increments (Chao et al., 2018). So, over the last forty years, a vast number of pricing models have been proposed as an alternative to the classic Black-Scholes approach.

One of the most popular extensions of the classical Black-Scholes model is to allow the volatility to be a stochastic process (Alòs and Yang, 2014).

The Black-Scholes model assumed that the volatility of the underlying security was constant, while stochastic volatility models categorized the price of the underlying security as a random variable or more general, a stochastic process. In its turn, the dynamics of this stochastic process can be driven by some other process (commonly by a Brownian motion) (Thao, H.T.P. and Thao, T.H., 2012). In a stochastic volatility model, the volatility changes randomly according to stochastic processes. This additional random source helps to partially explain why options with different strikes and maturities have different implied volatilities,

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which have been observed in market prices.

At about the same time as the stochastic volatility models were developed researchers argued that the bad fitting to real data was caused by the path continuity of the price process. Thus, the resulting model may have difficulties fitting financial data exhibiting large fluctuations. The necessity of taking into account large market movements and a great amount of information arriving suddenly (i.e., a jump) led researchers to propose models with jumps (Florescu et al., 2014).

Studies by Bates (Bates, 1991; Bates 1996) indicate that stochastic volatility and jumps are both features of the real world process and both effects are reflected in option prices (Albanese and Kuznetsov, 2005). Further, to capture jumps or discontinuities, fluctuations in this paper we use the Bates model as a base for pricing options.

In addition, extensive empirical studies show that the distributions of the logarithmic returns of financial asset usually exhibit properties of self-similarity and long-range dependence in both auto-correlations and cross-correlations. Since the fractional Brownian motion has the two important properties (self-similarity and long-range dependence), it has the ability to capture the behavior of underlying asset (stock) price (Podobnik et al., 2009; Pan and Zhou, 2017; Carbone et al., 2004; Wang, 2010a; Wang, 2010b; Wang, 2012; Sónia et al., 2008; Carbone, 2004). Further, to capture jumps or discontinuities, fluctuations and to take into account the long memory property, combination of jumps and fractional stochastic volatility is introduced in this paper. And due to the non-Markovian nature of the fractional Brownian Motion driven stochastic volatility, and market participants we use Monte-Carlo simulation along with a variance reduction technique (antithetic variates) in this article.

The rest of this paper is organized as follows. In Section 2, we start with a brief overview of stochastic volatility model of Heston. Then we show how Bates extend the Heston's model to models that involve jumps. In Section 3, briefly discusses the properties of fractional diffusion and its applications to option pricing. Then we present our approximative fractional stochastic volatility Model, which is mixed with a jump-diffusion model of market dynamics and we call it FSVJD model. Further this section is a step-by-step introduction to our method for exact simulation. And we derive a generic pricing PDE that attains an explicit semi-closed form solution for our FSVJD. As a measure long-range dependence, Hurst exponent Estimation is presented in Section 4. We use the different techniques to calibrate it. In Section 5, we calibrate our option pricing model to data obtained from the real market, namely we use daily data for S&P 500 Options. We give some numerical results and compare our model with Heston and Bates models. Section 6 draws the concluding remarks.

2. Sochastic volatility and Jump-Diffusion process

An extension to the assumption of constant volatility is to allow time dependence of volatility of the form $\sigma = \sigma(t)$.When taking the term structure into account one still can't account for the fact that different strikes give different implied volatilities. Dupire (Dupire, 1994) proposed a local volatility model, where volatility is both time and state dependent. He showed that it was possible to find $\sigma = \sigma(S_t, t)$ that accounts for the dynamics of the whole volatility surface (Clark, 2012).

Perhaps a more realistic assumption is that volatility is random in its behavior. The most popular model in this case is the one by Steven Heston (Heston, 1993). In 1993 Heston proposed a model where the volatility itself follows a random process, a so called square-root process. Under the risk-neutral measure Q, the model takes the following form:

$$dS_t = S_t(rdt + \sqrt{V_t}dW_t^1) \quad (1)$$

$$dV_t = -\kappa(V_t - \theta)dt + \sigma_v\sqrt{V_t}dW_t^2 \quad (2)$$

$$dW_t^1 dW_t^2 = \rho dt, \quad (3)$$

where S_t denotes the stock price, r is the risk-neutral rate of return, and W_t^1 and W_t^2 are two correlated Brownian motions under the risk-neutral measure, which are assumed to be stochastically independent under the original model. V_t represents a long term variance, θ is the long term mean of V_t, κ denotes the speed of reversion and the last parameter σ_v denotes volatility of V_t.

Another interesting approach in option pricing is the inclusion of jump processes. To improve flexibility of models and to enhance market calibration, many academics and professionals suggested a jump-diffusion modification of the stock price process S_t. Robert C. Merton in 1976 (Merton, 1976) was the first one who introduced a model to utilize jump-diffusion processes in finance. Further the Bates model was introduced by David Bates (Bates, 1996) in his 1996 paper and is an extension of the Heston model to include jumps in the stock price process. Bates referenced previous research showing that asset price variance is not constant—a theoretical validation for the SV model - and that asset price sample paths sometimes involve discontinuous jumps - a theoretical validation for the jump-diffusion model - and combined the two into the SVJD model (Kitchens, 2014). The model has the following risk-neutral dynamics defining the evolution of S_t:

$$dS_t = S_t((r-\lambda\mu_J)dt + \sqrt{V_t}dW_t^1) + J_tS_{t-}dN_t, \quad (4)$$

$$dV_t = -\kappa(V_t-\theta)dt + \sigma_v\sqrt{V_t}dW_t^2, \quad (5)$$

$$dW_t^1 dW_t^2 = \rho dt, \quad (6)$$

Where volatility process V_t is the same as that in the Heston model and the driving Brownian motions in the two processes have an instantaneous correlation equal to ρ. Under the notation S_{t-} we understand $lim_{\tau\to t\_} S_t$. The process N_t represents a Poisson process under the risk neutral measure, with jump intensity λ. The percentage jump size of the stock price is dictated by the random variable J, with

$$1 + J \sim \log - \text{normal}(\mu_S, \sigma_S^2) \quad (7)$$

where the relationship between μ_S and μ_J is given by

$$\mu_J = \exp\left\{\mu_S + \frac{\sigma_S^2}{2}\right\} - 1 \quad (8)$$

3. Fractional Stochastic Volatility with Jump Diffusion (FSVJD) Model

Comte and his colleagues, by the introduction of fractional noises, generalized the classical Heston model to account for long memory features of stochastic volatility (Comte and Renault, 1998; Comte et al., 2001). This technique allows to explain some option pricing puzzles such as steep volatility smiles in long term options and co-movements between implied and realized volatility (Bezborodov et al., 2016). Therefore to take into account the long memory property, and to get fluctuations form financial markets, it is suitable to apply the mixed fractional Brownian motion to take fluctuations from financial asset (see: Xiao et al., 2012; El-Nouty, 2003; Foremski et al., 2014). The mixed fractional Brownian motion is a family of Gaussian processes that is a linear combination of Brownian motion and fractional Brownian motion (Shokrollahi and Kılıçman, 2014).

In the mixed fractional Brownian motion, we will consider a process like this Instead of the usual fractional Brownian motion:

$$B_t = \int_0^t (t-s)^{H-1/2} dW_s \quad (9)$$

where H is a long-memory parameter ranging in [0, 1], and known as the Hurst exponent. If $H = 1/2$, then B_t is the usual standard Brownian motion. The process has a long memory for $H > 1/2$. Thus we consider only values of the Hurst parameter in this range (Sobotka et al., 2016). Moreover, we can approximate Bt by:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-1/2} dW_s \quad (10)$$

such that B_t^ε converges to B_t as ε tends to 0. The use of approximation B_t^ε instead of B_t provides several advantages. Firstly, there is no arbitrage opportunity under the approximative model dynamics for a wide class of simple and self-financing portfolios. Secondly, if we drive the process of dv_t as fractional process, we can use a standard Ito stochastic calculus instead of more advanced mathematical techniques for derivation of pricing PDE's.

Therefore here we are going to derive a general valuation PDE for stochastic volatility models without jumps and then generate the characteristic functions and at the end to consider jumps, we will add a characteristic function of a compound, compensated Poisson process to our model. Hence To describe approximative fractional approaches, we start with self-financing portfolio[1] concept. Let $X = X_t$ be the value of self-financing portfolio x at time t. Let x be delta and vega hedged (i.e. $\frac{\partial X}{\partial S_t} = 0, \frac{\partial X}{\partial v_t} = 0$) and let it consist of one option priced $V = V(S_t, v_t, t)$, $(-\Delta)$ units of the underlying stock with a price $S = S_t$ and $(-\Delta_1)$ units of another option with $V_1 = V_1(S_t, v_t, t)$. Then the portfolio value is determined by the following expression:

$$X = V - \Delta S - \Delta_1 V_1 \tag{11}$$

The portfolio is self-financing and thus a change in its value is given by:

$$dX = dV - \Delta dS - \Delta_1 dV_1 \tag{12}$$

Using Ito lemma, we can derive expressions for differentials dV and dV_1,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v_t}dv_t + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2}dt + \rho v_t \beta S \frac{\partial^2 V}{\partial v_t \partial S}dt, \tag{13}$$

$$dV_1 = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial S}dS + \frac{\partial V_1}{\partial v_t}dv_t + \frac{1}{2}v_t S^2 \frac{\partial^2 V_1}{\partial S^2}dt + \frac{1}{2}v_t\beta^2\frac{\partial^2 V_1}{\partial v_t^2}dt + \rho v_t \beta S \frac{\partial^2 V_1}{\partial v_t \partial S}dt, \tag{14}$$

Having explicitly expressed dV and dV_1 and hedging assumptions, we substitute the differentials into equation:

$$dX = \left(\frac{\partial V}{\partial t} + \frac{1}{2}v_t S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2} + \rho v_t\beta S\frac{\partial^2 V}{\partial v_t\partial S}\right)dt - \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}v_t S^2\frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}v_t\beta^2\frac{\partial^2 V_1}{\partial v_t^2} + \rho v_t\beta S\frac{\partial^2 V_1}{\partial v_t\partial S}\right)\Delta_1 dt = Adt - B\Delta_1 dt. \tag{15}$$

[1] The change in portfolio value Π is given only by changes in prices of the underlying assets for constant positions. And we cannot withdraw nor add funds to the portfolio.

we assume that there is a unique risk-free rate which we denote by r. We also utilize values of hedging parameters Δ, $\Delta 1$.

$$Adt - B\Delta_1 dt = r(V - \Delta S - \Delta_1 V_1)dt \tag{16}$$

$$A - B\Delta_1 = r(V - \Delta S - \Delta_1 V_1) \tag{17}$$

$$\frac{A - rV + \frac{\partial V}{\partial S}rS}{\frac{\partial V}{\partial v_t}} = \frac{B - rV_1 + \frac{\partial V_1}{\partial S}rS}{\frac{\partial V_1}{\partial v_t}} \tag{18}$$

Each side of Eq.(18) depends either on $V = V(S_t, v_t, t)$ or $V_1 = V_1(S_t, v_t, t)$. Both sides have to be equal to some function $g = g(S_t, v_t, t)$. In our case, we will closely follow [Gatheral, (2006)] and without loss of generality we set $g = -(\alpha - \varphi\beta\sqrt{v_t})$, where according to the Capital Asset Pricing Model (CAPM), φ represents the market price of volatility risk. As we are interested in the price of option V, we use just the left-hand side of Eq.(18). We also express the equation in terms of logarithm of the stock price, $x_t = ln(S)$, rather than S.

$$A - rV + \frac{\partial V}{\partial x_t}rx_t = -(\alpha - \varphi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t} \tag{19}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}v_t S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2} + \rho v_t\beta S\frac{\partial^2 V}{\partial v_t\partial S} - rV + \frac{\partial V}{\partial S}rS = -(\alpha - \varphi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t} \tag{20}$$

To simplify the last equation, we substitute $\tau = T - t$, where T is the time to maturity of option V.

We also express the equation in terms of logarithm of the stock price, $x_t = ln(S)$, rather than S.

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 V}{\partial x_t^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial V}{\partial x_t} + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2} + \rho v_t\beta\frac{\partial^2 V}{\partial v_t\partial x_t} - rV = -(\alpha - \varphi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t} \tag{21}$$

To obtain unique option prices, we chose the risk-neutral drift of dv_t, defined as $\hat{\alpha} = \alpha - \varphi\beta\sqrt{v_t}$, which rules out φ from our equations.

$$-\frac{\partial V}{\partial \tau}+\frac{1}{2}v_t\frac{\partial^2 V}{\partial x_t^2}+\left(r-\frac{1}{2}v_t\right)\frac{\partial V}{\partial x_t}+\frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2}+\rho v_t\beta\frac{\partial^2 V}{\partial v_t\partial x_t}-rV+\alpha\frac{\partial V}{\partial v_t}=0 \tag{22}$$

Price of a call option has to satisfy Eq.(22) with initial condition that is given by the pay-off function of the call option: $V_{call}=(ST\ -\ K)^+$. The price can be also expressed as an expectation of the discounted pay-off:

$$V_{call}(\tau,K)=e^{-r\tau}E[(ST-\ K)^+]=S_tP_1(x_t,v_t,t)-e^{-r\tau}KP_2(x_t,v_t,t)=e^{x_t}P_1(x_t,v_t,t)-e^{-r\tau}KP_2(x_t,v_t,t) \tag{23}$$

We can substitute Eq.(23) for V_{call} in Eq.(22). For $K=0$, $S_t=1$, we obtain the PDE with respect to *P* only.

$$-\frac{\partial P_1}{\partial \tau}+\frac{1}{2}v_t\frac{\partial^2 P_1}{\partial x_t^2}+\left(r-\frac{1}{2}v_t\right)\frac{\partial P_1}{\partial x_t}+\frac{1}{2}v_t\beta^2\frac{\partial^2 P_1}{\partial v_t^2}+\rho v_t\beta\frac{\partial^2 P_1}{\partial v_t\partial x_t}-rP_1+\left(\alpha-\varphi\beta\sqrt{v_t}\right)\frac{\partial P_1}{\partial v_t}=0 \tag{24}$$

Following similar arguments, we are able to retrieve the PDE for P_2 by setting $S_t=r=0, K=-1$.

$$-\frac{\partial P_2}{\partial \tau}+\frac{1}{2}v_t\frac{\partial^2 P_2}{\partial x_t^2}+\left(r-\frac{1}{2}v_t\right)\frac{\partial P_2}{\partial x_t}+\frac{1}{2}v_t\beta^2\frac{\partial^2 P_2}{\partial v_t^2}+\rho v_t\beta\frac{\partial^2 P_2}{\partial v_t\partial x_t}+\alpha\frac{\partial P_2}{\partial v_t}=0 \tag{25}$$

Instead of solving the system of two PDEs Eq.(24)-Eq.(25) directly, we express characteristic functions of the log-price at maturity *T*. After characteristic functions $f_j=f_j(\varphi,\tau)$ for j $=1,2$, are known, we can easily obtain P_j using the inverse Fourier transform.

$$P_j=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{w^{i\varphi\ln(K)}f_j}{i\varphi}\right)d\varphi \tag{26}$$

As in the original paper by Heston (Heston, 1993), we assume that characteristic functions f_j are of the following form:

$$f_j=\exp\{C_j(\tau,\varphi)+D_j(\tau,\varphi)v_0+i\varphi x\} \tag{27}$$

Firstly, we substitute assumed expression Eq.(27) for

$$-\left(\frac{\partial C_1}{\partial \tau}+v_t\frac{\partial D_1}{\partial \tau}\right)f_1+\rho v_t\beta i\varphi D_1f_1-\frac{1}{2}v_t\varphi^2+\frac{1}{2}v_t\beta^2D_1^2f_1+\left(r+\frac{1}{2}v_t\right)i\varphi f_1+(\alpha-\rho\beta v_t)D_1f_1=0 \tag{28}$$

Instead of a general drift of variance process dv_t, we assume a linear drift term with respect to v_t, i.e. $\alpha(S_t,v_t,t)=\theta+\overline{\alpha}(S_t,,t)v_t$. After rearranging terms with C_t, D_t and factoring out v_t we obtain:

$$v_t\left[-\frac{\partial D_1}{\partial \tau}+\rho\beta i\varphi D_1-\frac{1}{2}\varphi^2+\frac{1}{2}\beta^2D_1^2+\frac{1}{2}i\varphi+(\overline{\alpha}+\rho\beta)D_1\right]-\frac{\partial C_1}{\partial \tau}+ri\varphi+\theta D_1=0 \tag{29}$$

Since we assume that $v_t>0$, *so* we deduce that

$$\frac{\partial D_1}{\partial \tau}=\rho\beta i\varphi D_1-\frac{1}{2}\varphi^2+\frac{1}{2}\beta^2D_1^2+\frac{1}{2}i\varphi+(\overline{\alpha}+\rho\beta)D_1 \tag{30}$$

$$\frac{\partial C_1}{\partial \tau}=ri\varphi+\theta D_1 \tag{31}$$

Following the same steps, one can obtain a system of equations for f_2 as well. Therefore characteristic functions f_j defined by Eq.(27) have to satisfy the following system of four differential equations:

$$\frac{\partial D_1}{\partial \tau}=\rho\beta i\varphi D_1-\frac{1}{2}\varphi^2+\frac{1}{2}\beta^2D_1^2+\frac{1}{2}i\varphi+(\overline{\alpha}+\rho\beta)D_1 \tag{32}$$

$$\frac{\partial D_2}{\partial \tau}=\rho\beta i\varphi D_2-\frac{1}{2}\varphi^2+\frac{1}{2}\beta^2D_2^2+\frac{1}{2}i\varphi+\overline{\alpha}D_2 \tag{33}$$

$$\frac{\partial C_j}{\partial \tau} = ri\varphi + \theta D_j \quad (34)$$

And with respect to the initial condition:

$$C_j(0,\varphi) = D_j(0,\varphi) = 0 \quad (35)$$

We can obtain the first two equations which are known as Riccati equations. And after that we can solve the last two ODE's by a direct integration. We will rewrite equations Eq.(32) and Eq.(33) as following form:

$$\frac{\partial D_1(\tau,\varphi)}{\partial \tau} = A_j D_j^2 + B_j D_j + K_j, \quad (36)$$

Let us also denote:

$$\Delta_j = \sqrt{B_j^2 - 4A_jK_j}; \;\; Y_j = \frac{-B_j+\Delta_j}{2A_j}; \;\; g_j = \frac{B_j-\Delta_j}{B_j+\Delta_j}$$

Thus our solution would be:

$$D_j(\tau,\varphi) = \frac{Y_j\left(1-e^{\Delta_j\tau}\right)}{1-g_je^{\Delta_j\tau}} \quad (37)$$

In the next step, we integrate the right-hand side of Eq.(34) to express C_j.

$$C_j(\tau,\varphi) = ri\varphi\tau + \theta\int_0^{\tau} D_j(t,\varphi)dt = ri\varphi\tau + \theta\int_0^{\tau}\frac{Y_j\left(1-e^{\Delta_j\tau}\right)}{1-g_je^{\Delta_j\tau}}dt = ri\varphi\tau + \theta Y_j\left[\tau + \int_0^{\tau}\frac{(g_j-1)e^{\Delta_j\tau}}{1-g_je^{\Delta_j\tau}}dt\right] = ri\varphi\tau + \theta Y_j\tau - \theta Y_j\frac{g_j-1}{\Delta_j g_j}\ln\left(\frac{1-g_je^{\Delta_j\tau}}{1-g_j}\right) = ri\varphi\tau + \theta Y_j\tau - \frac{\theta}{A}\ln\left(\frac{1-g_je^{\Delta_j\tau}}{1-g_j}\right) \quad (38)$$

For diffusion stochastic volatility models, we obtain characteristic functions in the form of:

$$f_j(\tau,\varphi) = \exp\{C_j(\tau,\varphi) + D_j(\tau,\varphi)v_0 + i\varphi x\} \quad (39)$$

$$f_j^{(Heston)}(\tau,\varphi) = \exp\{C_j(\tau,\varphi) + D_j(\tau,\varphi)v_0 + i\varphi x\} \quad (40)$$

where for $j = 1,2$:

$$C_j(\tau,\varphi) = ri\varphi\tau + \frac{\theta}{\sigma_v^2}\left[(b_j + \rho\sigma_v\varphi i + \Delta_j)\tau - 2\ln\left(\frac{1-g_je^{\Delta_j\tau}}{1-g_j}\right)\right], \quad (41)$$

$$D_j(\tau,\varphi) = \frac{b_j+\rho\sigma_v\varphi i+\Delta_j}{\sigma_v^2}\left(\frac{1-e^{\Delta_j\tau}}{1-g_je^{\Delta_j\tau}}\right), \quad (42)$$

$$g_j = \frac{b_j - \rho\sigma_v\varphi i + \Delta_j}{b_j - \rho\sigma_v\varphi i - \Delta_j}, \quad \Delta_j = \sqrt{(\rho\sigma_v\varphi i - b_j)^2 - \sigma_v^2(2u_j\varphi i - \varphi^2)}, \quad u_1 = \frac{1}{2}, \; u_2 = -\frac{1}{2}, \quad \theta = k\bar{v}, \; b_1 = k - \rho\sigma_v, \; b_2 = k.$$

In case of the models with jumps (Bates model), we also need to include a characteristic function of a compound, compensated Poisson process, denoted by ψ.

$$f_j^{(Bates)}(\tau,\varphi) = \exp\{C_j(\tau,\varphi) + D_j(\tau,\varphi)v_0 + i\varphi x + \Psi(\varphi)\tau\}, \quad (43)$$

Thus in a model with jumps we must add the following function to other functions:

$$\Psi(\varphi) = -\lambda i\varphi\left(e^{\alpha+\gamma^2/2} - 1\right) + \lambda(e^{i\varphi\alpha-\varphi^2\gamma^2/2-1}) \quad (44)$$

To consider long memory we add Hurst parameter to Bates model to have advantage of jumps and long memory together. In this case by adding the Hurst parameter to the Bates model we reach to the characteristic functions of our FSVJD model in the following form:

$$f_j^{(FSVJD)}(\tau,\varphi) = \exp\{C_j(\tau,\varphi) + D_j(\tau,\varphi)v_0 + i\varphi x + \Psi(\varphi)\tau\}, \quad (45)$$

where for $j = 1,2$ and $\tau = T - t$:

$$C_j(\tau,\varphi) = ri\varphi\tau + \theta Y_j\tau - \frac{2\theta}{\beta^2}\ln\left(\frac{1-g_j e^{\Delta_j\tau}}{1-g_j}\right), \quad (46)$$

$$D_j(\tau,\varphi) = Y_j\left(\frac{1-e^{\Delta_j\tau}}{1-g_j e^{\Delta_j\tau}}\right), \quad (47)$$

$$\Psi(\varphi) = -\lambda i\varphi\left(e^{\alpha+\gamma^2/2} - 1\right) + \lambda(e^{i\varphi\alpha-\varphi^2\gamma^2/2-1}) \quad (48)$$

$$Y_j = \frac{b_j - \rho\beta\varphi i + \Delta_j}{\beta^2}$$

$$g_j = \frac{b_j - \rho\beta\varphi i + \Delta_j}{b_j - \rho\beta\varphi i - \Delta_j}, \quad \Delta_j = \sqrt{(\rho\beta\varphi i - b_j)^2 - \beta^2(2u_j\varphi i - \varphi^2)}, \quad \beta = \sigma_v \varepsilon^{H-1/2}\sqrt{v_t}, \quad u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad \theta = k\bar{v}, \; b_1 = k - (H - 1/2)\sigma_v\varphi_t^{\varepsilon}, \; b_2 = k - (H - 1/2)\sigma_v\varphi_t^{\varepsilon}.$$

4. Estimation of the Hurst exponent

We utilize several techniques to measure long-range dependence (LRD) in a given time-series. These techniques are well developed and implemented in various programming frameworks. In this section, we will focus on estimation of the Hurst exponent which is related to α,

$$\alpha = 2H - 1. \quad (49)$$

If we recall, $\alpha \in (0,1)$, one can easily see that for LRD processes H takes values from $1/2$ to 1 (Beran, 1994).

We employed synthetic data to find the best Hurst estimation method along these five techniques: Aggregate Variance method, Geweke-Porter-Hudak estimator, Higuchi method, Peng method, Periodogram analysis, and Rescaled range analysis. Which are introduced in (Beran, 1994; Geweke and Porter-Hudak, 1983; Higuchi, 1988; Hurst, 1951; Peng, et al., 1994).

To estimate the Hurst exponent from volatility data, our initial time series would be formed out of the increments of (simulated) volatility, i.e. the i_{th} element would be:

$$D_i = v_{t_i} - v_{t_{i-1}} \quad (50)$$

for $i = 1, 2, \ldots, N - 1$ while having N observations of the realized or simulated volatility.

To test different estimators we have simulated 10000 sample paths of five long-range dependence processes driven by fractional Brownian motions. Each process was simulated with different value of the Hurst exponent. The simulated processes follow a path wise SDE, where $\kappa = 2$, $\bar{v} = 0.1$, $\xi = 3$ are fixed and

$$H \in [0.60, 0.65, 0.70, 0.75, 0.80, 0.85] \quad (51)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^2 \quad (52)$$

Table1. Average estimates of Hurst parameter and Average absolute relative errors of estimates

	Aggregate Variance method	Geweke-Porter-Hudak estimator	Higuchi method	Peng method	Periodogram analysis	Rescaled range analysis
H=0.60	0.6082 (0.0011)	0.5988 (0.0009)	0.6008 (0.0005)	0.6082 (0.0009)	0.5887 (0.0056)	0.5946 (0.0011)
H=0.65	0.6448 (0.0005)	0.6591 (0.0010)	0.6482 (0.0007)	0.6468 (0.0007)	0.6475 (0.0010)	0.6448 (0.0008)
H=0.70	0.7096 (0.0012)	0.7015 (0.0008)	0.7064 (0.0019)	0.6951 (0.0005)	0.7156 (0.0032)	0.6931 (0.0013)
H=0.75	0.7403 (0.0007)	0.7702 (0.0065)	0.7552 (0.0009)	0.7469 (0.0011)	0.7403 (0.0015)	0.7512 (0.0012)
H=0.80	0.8089 (0.0010)	0.7902 (0.0010)	0.8022 (0.0006)	0.8041 (0.0008)	0.8012 (0.0003)	0.7905 (0.0009)
H=0.85	0.8543 (0.0012)	0.8422 (0.0009)	0.8551 (0.0007)	0.8513 (0.0005)	0.8352 (0.0049)	0.8434 (0.0011)
absolute errors (AAE)	%0.76	%1.54	%0.96	%0.56	%1.1	%1.01

After synthetic data were generated, we used procedures to obtain an estimate of the Hurst parameter alongside variance of the estimates. The most satisfying results were obtained using the Peng method which has the lowest average error.

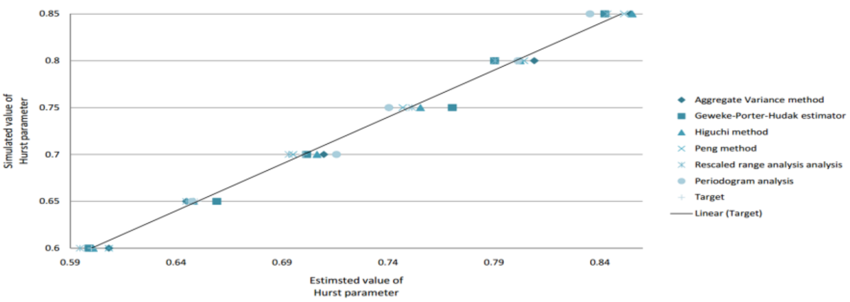

Figure1. Estimations of the Hurst parameter

5. Real Market Calibration

To calibrate models from the S&P 500 option market, at first we need to calibrate the parameters of our model. The model calibration is formulated as an optimization problem. The aim is to minimize the pricing errors between the model prices and the market prices for a set of traded options. A common approach to measure these errors is to use the squared differences between market prices and prices returned by the model, this approach leads to the nonlinear least square method. Mathematically put, given a model and a parameter set $\widehat{\Phi}$, we choose $\widehat{\Phi}$ as

$$\widehat{\Phi} = \arg\min \sum_{i=1}^{N} w_i \left(C_i^{Market}(S_0, K_i, T_i, r) - C_i^{Model}(S_0, K_i, T_i, r, \Phi)\right) \quad (53)$$

We set weights as a function of the price spread. Because the closer quoted ask and bid price are, the more efficiently is the given contract priced. So we minimized the criteria using the following weight functions:

$$W_i^A = \frac{1}{|\delta|} \quad (54)$$

$$W_i^B = \frac{1}{\sqrt{|\delta|}} \quad (55)$$

$$W_i^C = \frac{1}{|\delta|^2} \quad (56)$$

Where δ is the price spread of the i^{th} call option.

For comparative purposes, we also compute some other measures of fit such as the root mean square root error (RMSE), the average absolute error as a percentage of the mean price (APE) and the average absolute error (AAE):

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|C_i^{Model}(0,\Phi) - C_i^{Market}(0)\right|^2}, \quad (57)$$

$$APE = \frac{\sum_{i=1}^{n}\left|C_i^{Model}(0,\Phi) - C_i^{Market}(0)\right|}{\sum_{i=1}^{n} C_i^{Market}(0)} \quad (58)$$

$$AAE = \frac{1}{n}\sum_{i=1}^{n}\left|C_i^{Model}(0,\Phi) - C_i^{Market}(0)\right| \quad (59)$$

For calibration, we use Option data on the S&P 500 index consists of 263 trading days ranging from February 28 2011 to February 28 2012. Which was obtained from the Chicago Board Options Exchange's historical data retailer, Market Data Express[2]. The option data is daily best bid and ask prices. In order to prepare the data for use, we needed the length in years of each option, so a new column was created that computed length of each option by subtracting the expiration date from the quote date and converting into years. For any given day, there are many different strike prices and expiration lengths offered for options. We estimate the risk-neutral parameters for each model by inverting option prices in the training sample and use these estimates to predict prices of options in the test sample. We use Local Search method for calibrating the parameters of Heston, Bates and FSVJD models. And alongside we utilize weights Eq.(54), Eq.(55) and Eq.(56) to find the best calibration result.

Table 2. Calibration error measure for different weights

Models	Error measure	W^A	W^B	W^C
Heston	RMSE	3.4521	3.1325	3.3585
	APE	0.1325	0.1158	0.1317
	AAE	2.8546	2.8343	3.7562
Bates	RMSE	3.2384	2.8923	2.9512
	APE	0.0853	0.0571	0.0989
	AAE	2.8646	2.0456	2.8664
FSVJD	RMSE	2.8453	2.2598	2.6288
	APE	0.0786	0.0368	0.0756
	AAE	1.8435	1.7568	2.2548

[2] The product is called Option and was purchased from www.marketdataexpress.com via a link on www.cboe.com.

According to these results we obtained the best fit for W^B. So we employed this weight in our parameter calibration. Table 3 contain parameter calibration results of all three models.

Table 3. Calibrated parameters of the Heston, Bates and FSVJD model.

	Heston	Bates	FSVJD
v_0	0.0069	0.0064	0.0084
κ	1.2865	5.5863	2.0588
θ	0.0575	0.0534	0.0975
σ_v	0.0775	0.0856	0.0739
ρ	0.9889	0.9889	0.991
λ	-	0.7546	0.9548
α	-	0.9523	1.0133
γ	-	0.2941	0
H	-	-	0.6225

Therefore, by these parameters we can calculate the price of a European option with our proposed model. We use Monte Carlo simulation to generate an unbiased estimator of the price of options. In addition, it provides a convenient framework with which to approximate jumps in asset price. When using Monte Carlo simulation, many sample paths of the state variables are generated and the payoff of the derivative is evaluated for each path. Discounting and averaging over all paths gives an estimator of the derivative price. The error in the Monte Carlo estimator can be calculated using the central limit theorem and converges to zero as the number of sample paths used increases (Broadie and Kaya, 2006). Further we use a variance reduction technique, named Antithetic Variate, to improve the Monte Carlo simulation. So that sample sizes can be reduced for a given Monte Carlo variance. We draw from [Poklewski-Koziell, 2009] for our treatment on Monte Carlo methods for the Heston, Bates and FSVJD models. After pricing options with our FSVJD model, to have a comparison with two other well-known stochastic models, Heston and Bates, we compute the option prices with these models too. And here in table 4 we can see the comparison using three error measures (RMSE, APE, AAE).

Table 4. Pricing errors for all three models

	Heston	Bates	FSVJD
RMSE	4.0254	3.8426	3.1578
APE	0.0845	0.0718	0.0395
AAE	3.2454	2.8574	2.2157

As we see the results, we obtained the best fit (in terms of all measured errors) for the FSVJD model. And it is clear that the fractional model significantly improves the performance of the stochastic option pricing model.

6. Conclusion

In this paper, we compared several optimization approaches to the problem of option market calibration. For the empirical study we chose a popular SV model, firstly introduced by Heston (1993), and a more up to date approximative Fractional Stochastic Volatility model (FSV). To improve the model and to have a closer results for option pricing and to consider the jumps in asset prices, we provide the Fractional Stochastic Volatility Jump-Diffusion model (FSVJD). Actually the FSVJD model is the Bates model which has a Fractional Brownian Motion, once the fractional Brownian motion has the two important properties (self-similarity and long-range dependence) and we think it can improve the famous Heston and Bates models. For this reason after the semi-closed form solution of a generic pricing PDE is derived, we computes the Hurst exponent as a measure of long range dependence. We employed synthetic data to find the best Hurst estimation method. The most satisfying results were obtained using the Peng method which has the lowest average error. To calibrate the parameters of models alongside S&P 500 index call options we chose the best weight to fit the data for calibration. Local search method was used in order to minimize the difference between the observed market prices and the model prices and actually calibrating parameters of our FSVJD model and also Heston and Bates models. Then we price the options. The numerical results of presented in table 4 show that the FSVJD model works well against Heston and Bates models. We believe the complexity of the FSVJD model opens space for fine tuning the global optimizers.

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