Kuhn-Tucker's Theorem - the Fundamental Result in Convex Programming Applied to Finance and Economic Sciences

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***Abstract*— The optimization problems are not so important now in the field of production. But in the minimization risk problems, in profits maximization problems, in Marketing Research, in Finance, they are completely actual. An important example is the problem of minimizing portfolio risk, demanding a certain mean return. The main mathematical tool to solve these problems is the convex programming and the main result is the Kuhn-Tucker Theorem. In this work that result mathematical fundaments, in the context of real Hilbert spaces, are presented.**

***Keywords*—** *Convex programming, Kuhn-Tucker's Theorem, Optimization.*

1. Introduction

As an application of convex sets separation theorems, in Hilbert spaces, a class of convex programming problems, where it is intended to minimize convex functionals subject to convex inequalities, is considered.

Note that

Definition 1.1

A Hilbert space is a complex vector space with inner product that, as a metric space, is complete.

The Hilbert spaces are designated in this paper by *H* or *I*. Remember that

Definition 1.2

An inner product, in a complex vector space *H*, is a strictly positive sesquilinear hermitian functional on *H*.

Observation

- In real vector spaces sesquilinear hermitian must be substituted by bilinear symmetric.
- The inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ belonging to *H*, by this order, is designated by $[\boldsymbol{x}, \boldsymbol{y}]$.
- The norm of a vector $\boldsymbol{x}$ will be given by $|\boldsymbol{x}| = \sqrt{[\boldsymbol{x}, \boldsymbol{x}]}$.
- The distance between two elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of *H* will be $d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|$.

Definition 1.3

A set $K \subset H$ is convex if and only if

$$\forall_{\boldsymbol{x},\boldsymbol{y}\in \boldsymbol{K}} \forall_{\theta\in[0,1]} \quad \theta\boldsymbol{x} + (1-\theta)\boldsymbol{y} \in K.$$

Definition 1.4

A functional *p* defined in *H* is convex if and only if

$$\forall_{\boldsymbol{x},\boldsymbol{y}\in \boldsymbol{K}} \forall_{\theta\in[0,1]} \quad p(\theta\boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta p(\boldsymbol{x}) + (1-\theta)p(\boldsymbol{y}).$$

Then the separation theorems in Hilbert spaces, geometric consequences of Hahn-Banach theorem,[1] fundamental to the sequence of this work are presented.

Theorem 1.1

Be *A* and *B* two convex sets in a Hilbert space *H*. If one of them, for instance *A* has at least an inner point and $(intA) \cap B = \emptyset$, so there is at least a non-null vector $\boldsymbol{v}$ such that

$$\sup_{x \in A}[\boldsymbol{v}, \boldsymbol{x}] \leq \inf_{y \in B}[\boldsymbol{v}, \boldsymbol{y}].$$

Theorem 1.2

Given a closed convex set *A* in a Hilbert space *H* and a point $\boldsymbol{x}_0 \in H$ not belonging to *A*, there is a non-null vector $\boldsymbol{v}$ such that

$$[\boldsymbol{v}, \boldsymbol{x}_0] < \inf_{x \in A}[\boldsymbol{v}, \boldsymbol{x}].$$

Theorem 1.3

Two closed convex subsets *A* and *B*, of a Hilbert space, each other finite distanced, that is: such that

$$\inf_{x \in A, y \in B}|\boldsymbol{x} - \boldsymbol{y}| = d > 0$$

may be strictly separated. That is: there is at least a $\boldsymbol{v} \in H$ for which

$$\sup_{x \in A}[\boldsymbol{v}, \boldsymbol{x}] < \inf_{y \in B}[\boldsymbol{v}, \boldsymbol{y}].$$

Theorem 1.4

Being *H* a finite dimension Hilbert space, if *A* and *B* are convex sets, not empty, disjoint so they can always be separated, that is: there is at least a non-null vector $\boldsymbol{v}$ such that

$$\sup_{x \in A}[\boldsymbol{v}, \boldsymbol{x}] \leq \inf_{y \in B}[\boldsymbol{v}, \boldsymbol{y}].$$

Finally an important property of the Hilbert spaces convex continuous functionals:

Theorem 1.5

A continuous convex functional in a Hilbert space has minimum in any limited closed convex set.

Demonstration:

If the space is of finite dimension, obviously the condition of the convexity for the set is not needed. In spaces of infinite dimension, note that if $\{x_n\}$ is a minimizing sequence, so, as the sequence is bounded, it is possible to work with a weakly convergent sequence and there is weak lower semi continuity, see for instance (1): $\underline{limf}(x_n) \geq f(x)$, calling f (.) the functional, where *x* is the weak limit, and consequently the minimum is *f*(*x*). As a closed convex set is weakly closed, *x* belongs to the closed convex set■.

2. Kuhn-Tucker's Theorem

From now on only real Hilbert spaces are considered.

Theorem 2.1 (Kuhn-Tucker)

Be $f(\boldsymbol{x}), f_i(\boldsymbol{x}), i = 1, \dots, n,$ convex functionals defined in a convex subset *C* of a Hilbert space.

Consider the problem

$$\min_{x \in C} f(\boldsymbol{x})$$
$$sub: f_i(\boldsymbol{x}) \leq 0, i = 1, \dots, n.$$

Be $\boldsymbol{x}_0$ a point where the minimum, supposed finite, is reached.

Suppose also that for each vector $\boldsymbol{u}$ in E_n (Euclidean space of dimension *n*), non-null and such that $u_k \geq 0$, there is a point $\boldsymbol{x}$ in *C* such that

$$\sum_{k=1}^{n} u_k f_k(\boldsymbol{x}) < 0 \qquad (2.1)$$

where u_k are the coordinates of $\boldsymbol{u}$.

Thus,

[1] **Theorem (Hahn-Banach)**
Be *p* a positively homogeneous convex functional defined in a real vector space *L* and L_0 a *L* subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition,

$$f_0(x) \leq p(x), \forall_{x \in L_0}$$

so exists an extension *f* of f_0 defined in *L*, linear, and such that

$$f(x) \leq p(x), \forall_{x \in L}. .$$

i) There is a vector $\boldsymbol{v}$, with non negative coordinates v_k, such that

$$\min_{\boldsymbol{x}\in C}\left\{f(\boldsymbol{x})+\sum_{k=1}^{n} v_k f_k(\boldsymbol{x})\right\} = f(\boldsymbol{x}_0)+\sum_{k=1}^{n} v_k f_k(\boldsymbol{x}_0) = f(\boldsymbol{x}_0), \qquad (2.2)$$

ii) For any vector $\boldsymbol{u}$ in E_n with non negative coordinates (it is also said: belonging to the positive cone of E_n)

$$f(\boldsymbol{x})+\sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \sum_{k=1}^{n} v_k f_k\big(\boldsymbol{x}_0 \geq f(\boldsymbol{x}_0)\big) + \sum_{k=1}^{n} u_k f_k(\boldsymbol{x}_0). \qquad (2.3)$$

Demonstration:

Be the sets A and B in E_{n+1}:

$$A:\{\boldsymbol{y}=(y_0,y_1,\dots,y_n)\in E_{n+1}: y_0 \geq f(\boldsymbol{x}), y_k \geq f_k(\boldsymbol{x}) \text{for some } \boldsymbol{x} \text{ in } C, k = 1,\dots,n.\},$$

$$B:\{\boldsymbol{y}=(y_0,y_1,\dots,y_n)\in E_{n+1}: y_0 < f(x_0), y_i < 0, i=1,\dots,n.\}.$$

It is easy to verify that A and B are convex sets in E_{n+1}, disjoint.

So they can be separated, that is, it is possible to find $v_k, k = 0,1,\dots,n$ such that

$$\inf_{\boldsymbol{x}\in C} v_0 f(\boldsymbol{x})+\sum_{k=1}^{n} f_k(\boldsymbol{x}) \geq v_0 f(x_0) - \sum_{k=1}^{n} v_k|y_k|. \qquad (2.4)$$

As (2.4) must hold for any $|y_k|$, it is concluded that $v_k, k=1,\dots,n$, is non negative. In particular approaching $|y_k|$ from zero it is obtained

$$v_0 f(x_0)+\sum_{k=1}^{n} v_k f_k\,(\boldsymbol{x}) \geq v_0 f(x_0)$$

and as the $f_k(x_0)$ are non positive it follows that

$$\sum_{k=1}^{n} v_k f_k(x_0) = 0. \qquad (2.5)$$

Then it is shown that v_0 must be positive

In fact if the whole $v_k, k=1,\dots,n$ are zero, v_0 cannot be zero, and from $v_0 z_0 \geq v_0 y_0$ for any $y_0 < f(x_0) < z_0$, it follows that v_0 must be positive.

Supposing now that not all the v_k are zero, k=1, …, n, there is an $\boldsymbol{x}\in C$ such that $\sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) < 0$ (by hypothesis). But for any z_0 greater or equal than $f(\boldsymbol{x})$ it must be $v_0\big(z_0 - f(x_0)\big) \geq -\sum_{k=1}^{n} v_k f_k(x_0) > 0$, and so v_0 must be positive. So, after (2.4) and putting $V_k = \frac{v_k}{v_0}, k=1,\dots,n$ it is obtained

$$f(\boldsymbol{x})+\sum_{k=1}^{n} V_k f_k(\boldsymbol{x}) \geq f(x_0) = f(x_0)+\sum_{k=1}^{n} V_k f_k(x_0),$$

resulting in consequence the remaining conclusions of the theorem■.

Observation:

- A sufficient condition, obvious but useful, so that (2.1) holds is that there is a point $\boldsymbol{x}$ in C such that $f_i(\boldsymbol{x})$ is lesser than zero for each $i, i=1,\dots,n$.

Corollary 2.1 (Lagrange's Duality Theorem)

In the conditions of Kuhn-Tucker Theorem

$$f(x_0) = \sup_{\boldsymbol{u}\geq 0}\inf_{\boldsymbol{x}\in C}\left(f(\boldsymbol{x})+\sum_{k=1}^{n} u_k f_k(\boldsymbol{x})\right)$$

Demonstration:

$\boldsymbol{u} \geq 0$ means that the whole coordinates $u_k, k = 1, \dots, n$, of $\boldsymbol{u}$ are non negative. The result is a consequence of the arguments used in the Theorem of Kuhn-Tucker demonstration:

- For any $\boldsymbol{u} \geq 0$

$$\inf_{x \in C}\left(f(\boldsymbol{x}) + \sum_{k=1}^{n} u_k f_k(\boldsymbol{x})\right) \leq f(x_0)$$

$$+ \sum_{k=1}^{n} u_k f_k(x_0) \leq f(x_0).$$

- In particular for $u_k = v_k$

$$\inf_{x \in C}\left(f(\boldsymbol{x}) + \sum_{k=1}^{n} v_k f_k(\boldsymbol{x})\right) \geq f(x_0).$$

then resulting the conclusion ■.

Observation:

- This Corollary gives a process to determine the problem optimal solution.

- If the whole v_k in expression (2.3) are positive, x_0 is a point that belong to the border of the convex set determined by the inequalities.

- If the whole v_k are zero, the inequalities are redundant for the problem, that is: the minimum is the same as in the "free" problem (without the inequalities restrictions).

3. Kuhn-Tucker's Theorem for Inequalities in Infinite Dimension

In this section, the situation resulting from the consideration of infinite inequalities will be studied. A possible approach is:

- To consider a transformation $F(x)$ from a real Hilbert space H to L_2: space of the summing square functions sequences.

- To consider the positive cone $\wp$, in L_2, of the sequences which the whole terms are non-negative.

- To consider the negative cone $\aleph$, in L_2, of the sequences which the whole terms are non-positive.

- To formalize the problem of the minimization of the convex functional $f(x)$, constrained to $x \in C$ convex, as in section 2, and $F(x) \in \aleph$, supposing that $F(x)$ is convex.

Unfortunately the Kuhn-Tucker's theorem does not deal with this situation.

Similarly to the demonstration of Theorem 2.1 define

$$A = \{(y, z): y \geq f(x) \wedge z - F(x) \in \wp \text{ for any } x \in C\},$$

$$B = \{(y, z): y < f(x_0) \ \wedge \ z \in \aleph\},$$

where x_0 is a minimizing point, as before. But, now, A and B, even being disjoint, can not necessarily be separated if neither A nor B have interior points. And evidently $\aleph$ has not interior points.

Another way, in order to establish a generalization, may be:

- To consider a real Hilbert space I that encloses a **closed convex cone** $\wp$.

- Given any two elements $x, y \in I$, $x \geq y$ if $x - y \in \wp$.

It is a well defined order relation: if $x \geq y$ and $y \geq z, x - y \in \wp$ and $y - z \in \wp$; being $\wp$ a convex cone, $(x - y) + (y - z) \in \wp$, that is $x \geq z$.

- So $\wp$ may be given by $\wp = \{x \in I: x \geq 0\}$ and may be called **positive cone**.

- The **negative cone** $\aleph$ will be given by $\aleph = -\wp = \{x \in I: x \leq 0\}$.

Having as reference these order relation, it is possible to define a convex transformation in the usual way. If the cone $\aleph$ has a non-empty interior, a version of the **Kuhn-Tucker's theorem for infinite dimension inequalities** may be established.

Theorem 3.1 (Kuhn-Tucker in Infinite Dimension)

Call C a convex subset of a real Hilbert space H and $f(x)$ a real convex functional defined in C.

Be I a real Hilbert space with a convex closed cone $\wp$, with non-empty interior, and $F(x)$ a convex transformation from H to I – convex in relation with the order induced by the cone $\wp$.

Consider x_0, a minimizing of $f(x)$ in C, constrained to the inequality $F(x) \leq 0$.

Call $\wp^* = \{x: [x,p] \geq 0, \text{for any } p \in \wp\}$ - the dual cone.

Admit that given any $u \in \wp^*$ it is possible to determine x in C such that $[u, F(x)] < 0$.

So, there is an element v in the dual cone $\wp^*$, such that for x in C

$$f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)],$$

where u is any element of $\wp^*$.

Demonstration:

It is identical to the one of Theorem 2.1. Build A and B, subsets of $E_1 \times I$:

$$A = \{(a,y): a \geq f(x), y \geq F(x), \text{ for any } x \text{ in } C\},$$

$$B = \{(a,y): a \leq f(x_0), y \leq 0\}.$$

In the real Hilbert space $E_1 \times I$, these sets can be separated, since B has non-empty interior and $A \cap B$ has not any interior point of B. So it is possible to find a number a_0 and $v \in I$ such that, for any x in C, $a_0 f(x) + [v, F(x)] \geq$ $a_0 f(x_0) - [v, p]$ for any p in $\wp$. As this inequality left side is lesser than infinite, it follows that $[v, p] \geq 0$, for any $p \in \wp$ and so $v \in \wp^*$.

The remaining demonstration is a mere copy of the Theorem 2.1' s■.

Observation:

- Through subtle, although conceptually complicated, generalization of Kuhn-Tucker's theorem it was possible to present the mathematical fundaments of Kuhn-Tucker's theorem in infinite dimension. It was necessary to define very carefully the domains to be considered: the Hilbert spaces and the adequate cones. And this is a really challenging problem from the mathematical point of view.

There is also a version in infinite dimension for the Lagrange's Duality Theorem:

Corollary 3.1 (Lagrange's Duality Theorem in Infinite Dimension)

In the conditions of Kuhn-Tucker's Theorem in Infinite Dimension

$$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)]).$$

Observation:

- Also, as in the former section, this Corollary gives a process to determine the problem optimal solution.

4. Conclusions

Convex programming is a powerful tool to solve practical problems in various domains, namely in Operations Research, Economics, Management, etc. In its various branches – Linear Programming, Integer Programming, Quadratic Programming, Assignment Problems and even Dynamic Programming – it allows the mathematical modelling of a lot of practical problems allowing a better knowledge of them and their solution determination.

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