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The Relative Risk Aversion (RRA) Riddle

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Abstract:

Purpose: *Looking for a way to solve the risk aversion puzzle, this article develops a model that incorporates loss aversion into the state dependent recursive preferences.*
Design/Methodology/Approach: *In this model, the representative agent is restricted to be risk averse in all states of nature. In addition, I calibrate the model in selected developed countries.*
Findings: *The model succeeds in generating a relative risk aversion value below ten, but the generated stochastic discount factor fails in matching its implied values. Moreover, a representative agent who has the S-shaped loss averse preferences could not solve the model either.*
Practical implications: *Hence, it reveals another distinctive risk aversion riddle.*
Originality value: *A low risk aversion value is not the answer to the equity premium puzzle. Nevertheless, studying the effect of loss aversion on recursive preferences gives delightful insights and better understanding of many details to the problem in hand.*

Keywords: *Loss aversion, stochastic discount factor, state dependent preferences, equity premium puzzle.*

JEL Codes: *G10, G12.*

Paper type: *Research Paper.*

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1. Introduction

Behavioral finance proposed various modifications to preferences such as loss aversion and disappointment aversion to account for high equity premium levels or the high relative risk aversion entailed to explain these levels. Loss aversion refers to a situation in which the individual's sensitivity to losses is more than his sensitivity to gains. As per Kahneman and Tversky (1979) and Tversky and Kahneman (1992), a loss averse agent is risk averse in the positive domain and risk lover in the negative domain. The positive and negative domains are to be determined by a reference point.

Thus, loss aversion leads to an asymmetry in the risk aversion behavior of the agents. While Andries (2012) preferences of loss aversion, unlike the S-shaped utility function of Kahneman and Tversky, implicitly restricts the representative agent to be always risk averse. Loss aversion has extensive experimental evidence.

Various preferences ordering emerged in the literature to explain the equity premium puzzle. Epstein-Zin recursive preferences are the only preferences which separate relative risk aversion and elasticity of intertemporal substitution parameters. Consequently, these preferences could generate low and sTable risk-free rates along with high and volatile stock returns.

In addition, recursive preferences allow the stochastic discount factor to vary across states of nature. Furthermore, state dependent recursive preferences allow relative risk aversion and elasticity of intertemporal substitution to take different values in different states which further increases the volatility of the stochastic discount factor. A high variability of the stochastic discount factor is required to match asset market historical data (Hansen and Jagannathan, 1991). Melino and Yang (2003) stress the need for state dependent recursive preferences to better match asset market characteristics.

Adding loss aversion to the state dependent recursive preferences model retains the difference in the representative agent's behavior across states to loss aversion. Throughout the literature, loss aversion succeeded in producing low values of relative risk aversion which motivated this article to examine if it can improve on the state dependent recursive preferences model. No study has examined the case of loss aversion either in the context of Mehra-Prescott settings nor with state dependent preferences up to the author's knowledge.

The central argument for the plausibility of a low value of relative risk aversion relies on the estimated price of risk computed by Friend and Blume (1975). Friend and Blume estimated relative risk aversion to be at least two. Although they did not specify an upper bound, it has been considered in the asset pricing literature that a value of relative risk aversion below ten is plausible. Mehra and Prescott (1985), in studying the United States asset market during the period 1890-1978, restricted relative risk aversion to be between zero and ten.

The literature is divided regarding the accepTable values of relative risk aversion. Some studies consider a reasonable relative risk aversion value to be a value lower than ten. In contrast, other studies demonstrate that relative risk aversion higher than ten might be accepTable. This restriction, for the reasonable values of relative risk aversion, generated the equity premium puzzle.

This paper adds loss aversion to the model of state dependent recursive preferences. Moreover, it calibrates the model in selected developed countries and investigates if loss aversion can explain the equity premium puzzle with a reasonable relative risk aversion value. On one hand, loss aversion lowers the relative risk aversion of the representative agent to be within accepTable limits while matching equity premium historical data.

On the other hand, the model's stochastic discount factor is not volatile enough. It fails to match its values implied by historical data. Besides, it violates Hansen-Jagannathan bound. This reveals another puzzle which I call the risk aversion riddle.

My model differs from Andries (2012) loss averse preferences in that it is a discrete time model. Using discrete instead of continuous time fits better the investigated data. Another difference from Andries is that she uses the consumption growth and earnings to consumption ratios on five lagged periods as a state variable; instead, I adopt the simple Mehra-Prescott environment where the state variable is the real consumption per capita growth.

I follow Melino and Yang (2003) in my choice of real consumption per capita growth to be the state variable as it could generate a stationary recursive equilibrium and for comparability of results. Moreover, I solve for the value of relative risk aversion and elasticity of intertemporal substitution unlike the model of Andries for which she assumes a constant value for the relative risk aversion parameter.

The mainstream in the equity premium puzzle literature considers relative risk aversion greater than ten as implausible. One of the main critics of asset pricing models which attempted to explain the equity premium puzzle is the high relative risk aversion needed to explain equity premium historical data.

Hence, I introduce Andries' loss aversion setting to the state dependent recursive preferences of Melino and Yang (2003) to examine if a loss averse representative agent can explain asset market characteristics with a plausible value for relative risk aversion. I would like to explore the research question: Can combining loss aversion and state dependent recursive preferences provide an explanation for the equity premium puzzle with reasonable relative risk aversion and at the same time satisfy equity market characteristics? Answering this question enables us to explore if loss aversion can solve the elevated level of risk aversion concern within the state dependent preferences context, and whether at the same time it keeps matching all the market characteristics or in lowering the risk aversion leads to some violations.

2. Literature Review

The equity premium puzzle emerged due to the inability of asset pricing models to explain the six percent premium observed in the United States historical data. Mehra and Prescott (1985) found, in a competitive pure exchange economy using a standard utility function, that average equity real annual return should be a maximum of 0.4 percent more than that on the short-term debt. Consequently, investors must be implausibly risk averse for theoretical models to match equity premium levels.

Hansen and Jagannathan (1991) provided an alternative characterization of the equity premium puzzle. They investigated the restrictions imposed on the stochastic discount factor and standard deviation by asset market data. They found that, for any portfolio, the standard deviation-mean ratio of the stochastic discount factor should be at least equal to the Sharpe ratio, which is referred to as Hansen-Jagannathan bound. This bound imposes a challenge to a large class of asset valuation models.

High equity premium levels were recognized in developed countries other than the United States (Campbell, 1999; Mehra, 2003; Dimson, Marsh, and Staunton, 2011). Campbell (1999) considered the Morgan Stanley Capital International (MSCI) stock market quarterly data in Australia, Canada, France, Germany, Italy, Japan, the Netherlands, Sweden, Switzerland, the United Kingdom, and the United States. His sample has starting dates ranging from 1970.1 to 1982.2 and has ending dates which range from 1995.1 to 1996.4.

Quarterly equity premium had a maximum value of 10% in Switzerland and a minimum value of -1.6% in Italy. In addition, He examined annual data for Sweden (1920-1994), the United Kingdom (1919-1994) and the United States (1891-1995). Annual equity premium was equal to 4% in Sweden, 6% in the United Kingdom and 4.7% in the United States.

While Mehra (2003) calculated the annual equity premium levels in the United Kingdom (1947-1999), Japan (1970-1999), France (1973-1998) and Germany (1978-1997) to be 4.6%, 3.3%, 6.3% and 6.6%. Besides, Dimson, Marsh, and Staunton (2011) calculated the geometric and the arithmetic annual average equity premium levels relative to bills and to bonds running from 1900 to 2010.

Their sample included Australia, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, the Netherlands, New Zealand, Norway, South Africa, Spain, Sweden, Switzerland, the United Kingdom, and the United States. They also calculated the equity premium for Europe, the World and the World excluding the United States. Arithmetic equity premium relative to bills ranged between 4.6% in Denmark and 9.8% in Germany and Italy.

Given the observed high equity premium, economists investigated various explanations [2] for this puzzle. Behavioral models were one of the proposals developed to explain the puzzle or more generally the violations[3] of expected utility paradigm in the context of asset pricing.

Kahneman and Tversky (1979) and Tversky and Kahneman (1992) introduced prospect theory to the literature. Prospect theory defined a loss averse agent as the one who is more sensitive to losses than to gains, which creates asymmetry in his behavior. This differentiation did not exist before in standard asset pricing models; in which the domain is final states rather than gains and losses. Standard asset pricing models provide the same weight to gains and losses.

Individuals' sensitivity to reductions in their budget than increases is valid empirically. Kahneman and Tversky defined loss aversion coefficient as the ratio of the slopes of utility, under gains and under losses, with the slope of the utility under losses in the numerator. They found that loss aversion coefficient equals 2.25. Empirical estimates of loss aversion coefficient are usually around 2, meaning that the disutility of a loss is twice as much as the utility of the gain.

Benartzi and Thaler (1995) took a step further by combining the sensitivity to a loss with the frequency of monitoring wealth. This tendency is called myopic loss aversion. It changes the domain of the utility function to be wealth instead of consumption. Using simulations, they found that equity premium matched its historical level with a loss aversion coefficient of 2.77; if investors evaluate their portfolios annually.

In contrast, Barberis *et al.* (2001) showed that loss aversion alone cannot provide a complete description of the aggregate stock market. In this paper, they included financial wealth and consumption in the utility function.

They could explain the high means, excess volatility of returns as well as their low correlation with consumption growth using a traditional consumption-based asset pricing model of Lucas (1978). While Barberis and Huang (2009) combined loss

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[2] *Discussing all various explanations for the equity premium puzzle is out of the scope of this paper. For this purpose, the reader may refer to Kocherlakota (1996), Abel (1991), Cochrane and Hansen (1992), Heaton and Lucas (1995), Cochrane (1997), Campbell (1999 and 2001) and Mehra and Prescott (2003) among others.*

[3] *Using hypothetical choices, Kahneman and Tversky (1979) demonstrated substitution and independence axioms of expected utility theory are violated. The best-known counter example to expected utility theory is Allias paradox.*

aversion with narrow framing[4]. This combination could generate a high equity premium and a low and sTable risk-free rate when consumption is smooth and weakly correlated with the stock market. Moreover, they could do this for reasonable relative risk aversion parameters. But both Barberis *et al.* (2001) and Barberis and Huang (2009) used a power utility function.

The standard power utility is criticized for linking intertemporal substitution and relative risk aversion; being reciprocal of one another. Therefore, different modifications on the utility function were introduced to explain the puzzle. Epstein-Zin recursive preferences provide an adequate description of individuals' behavior.

It is the only type of preference which disentangles relative risk aversion and elasticity of intertemporal substitution. Consequently, it generates low and sTable risk-free rates along with high and volatile risky returns as empirically observed. Besides, recursive preferences are intertemporally consistent.

A state dependent Epstein-Zin preference addresses the idea of the agent's asymmetric behavior like behavioral models, yet in a different way. It allows preferences' parameters to vary across states of nature. Melino and Yang (2003) and Ross (2015) confirmed the importance of state dependence.

Melino and Yang (2003) studied state dependent recursive preferences in a Mehra-Prescott environment. Matching the first two moments of consumption real growth, the risk-free rate and market equity return, they could explain the United States equity premium levels with a relative risk aversion value lower than Mehra and Prescott. While Ross (2015) relied on state dependence to fully recover state prices.

Behavioral models combined with recursive preferences could explain better the equity premium puzzle (Barberis and Huang, 2009; Routledge and Zin, 2010; Andries; 2012). Routledge and Zin (2010) studied recursive preferences with generalized disappointment aversion. Rather, Andries (2012) developed a model of loss aversion with recursive preferences where agents are risk averse in both the domain of gains and the domain of losses.

Her model could produce lower risk-free rates compared to the standard recursive utility model. But the maximum equity premium that this model could generate is 3.39% which is still far from the 6.09% average historical level observed in the same period (1947-2010).

In this paper, I develop a model which incorporates loss averse preferences within the state dependent recursive preferences in the simple Mehra-Prescott environment

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[4] *Narrow framing refers to the setting where agents evaluate new gambles in isolation, separately from their other risks.*

and study its implications for selected developed countries. Theoretical framework, data, methodology and empirical results are presented in the following sections.

3. Theoretical Framework

This paper develops an asset pricing model in which the representative agent is loss averse and has state dependent recursive preferences. In the first subsection I introduce the model. Then I discuss the equilibrium in the second subsection.

3.1 The Model

My model incorporates the loss averse preferences of Andries (2012) in Melino-Yang state dependent preferences model. Unlike Andries, my model is a discrete time model. The only state variable is real per capita consumption growth rate, following the mainstream under the Consumption based Capital Asset Pricing Model, while Andries uses the consumption growth and earnings to consumption ratios on five lagged periods as a state variable instead.

In line with Andries (2012), the representative agent is risk averse in all states of nature. In contrast, the agent in Kahneman and Tversky's model displays loss aversion in the form of an S-shaped utility, being risk averse above and risk seeker below a reference point. I solved the model with Kahneman and Tversky S-shaped preferences as well.

The economy is a Lucas endowment frictionless economy in which real per capita consumption growth is Markov stationary, which is referred to as Mehra-Prescott environment. At time t, current consumption is known but future consumption levels are stochastic. There are two assets; a stock market index which corresponds to equity and a government short-term treasury bill as a proxy for the riskless asset. The equity share is completely traded. I assume the subjective discount factor β is constant.

I consider an infinitely lived representative agent, who receives utility from the consumption of a single good in each period. He has temporal Von Neumann Morgenstern state dependent recursive preferences[5]. The representative agent has recursive preferences as follows:

$$U_t(c_t, U_{t+1}) = \left(c_t^{\rho} + (\beta(\mu_t(U_{t+1}))^{\rho})\right)^{\frac{1}{\rho}} \qquad t = 1, \dots, \infty \qquad (1)$$

$0 \leq c_t \leq x_t, 0 < \beta < 1 \; and \; 0 \neq \rho < 1$

[5]*For further details and arguments about using recursive preferences, please refer to Kreps and Porteus (1979) as well as Epstein and Zin (1989).*

where c_t refers to the representative agent's real current consumption, x_t is the beginning-of-period wealth, β represents the agent's subjective discount factor, μ_t is the certainty equivalent function as given in equation 2 below, s_t is the state of nature and the elasticity of intertemporal substitution equals $1/(1-\rho)$ (A proof of this statement is provided in Appendix E).

$$\mu_t(U_{t+1}) = \begin{cases} \left(E_t\left(U_{t+1}^{\alpha_l}\right)\right)^{\frac{1}{\alpha_l}} & U_{t+1} < Ref_t \\ \left(E_t\left(U_{t+1}^{\alpha} * Ref_t^{\ \alpha_l-\alpha}\right)\right)^{\frac{1}{\alpha_l}} & U_{t+1} \geq Ref_t \end{cases} \tag{2}$$

With $0 \leq 1-\alpha_l \leq \infty, 0 \leq 1-\alpha \leq \infty$ and $\alpha_l < \alpha$

where E_t is the expectation conditional on time t information, $1-\alpha$ indicates the representative's agent relative risk aversion[6] with a current high state, and $1-\alpha_l$ represents the value of relative risk aversion when current state of nature is low[7]. The representative agent is risk averse in both states.

Relative risk aversion (hereafter risk aversion) and elasticity of intertemporal substitution are state dependent. States of nature are determined relative to the state variable. There exists a high state of nature and a low one. I use a simple threshold model to define the threshold between both states. I assume both states of nature are equally likely to happen with a symmetric transition matrix for the conditional probabilities.

The loss averse agent values consumption outcome relative to a reference point. It refers to the point below which the agent suffers dis-utility because of loss aversion. Deviations from this point define gains or losses. Following Andries (2012), the deterministic endogenous reference point is the agent's expectation of future consumption utility[8]. Whenever the agent's expectation of future consumption stream is less than this point, he suffers disutility. The log-linear specification for the reference point is given by equation 3.

$$Ref_t = e^{E_t[\log(U_{t+1})]} \tag{3}$$

[6] *I use the Arrow-Pratt measure of risk aversion.*

[7] *For a proof that relative risk aversion is $1-\alpha$ and $1-\alpha_l$ in the current high and low state respectively, the reader may refer to Epstein and Zin (1989).*

[8] *The reference point is conventionally set to be equal to the status quo, but Koszegi and Rabin (2006) argue that the reference point should be determined by the agent's expectation of outcomes. Expectations generally make better predictions in a situation when expectations and the status quo are different, a common situation in an economic environment.*

Proposition 1. *Given:*

(1) *A representative agent with recursive preferences* $U_t = f(c_t, \mu_t)$, *where* μ_t *is a certainty equivalence function.*
(2) *There are two states of nature: a low and a high one.*

Then, the representative agent is loss averse in the current low state of nature with an endogenously determined log linear reference point.

Proof: See Appendix A.

Hence, with only two states of nature, $U_{t+1} < Ref_t$ occurs when the current state of nature is low while $U_{t+1} \geq Ref_t$ is satisfied with a current high state. Representative agent's preferences have a pair wise certainty equivalence function μ_t as in equation 2; depending on whether the agent's value function is above or below this reference point. When the value function is greater than the reference point, utility is weighted by the term ${Ref_t}^{\alpha_l - \alpha}$. In this case the weight is equal to one. In Appendix B, I show the derivation of equation 2.

A state dependent risk aversion makes the slope of the certainty equivalent function different in each state. The ratio of the slopes of U_{t+1} in the certainty equivalence function; with the slope under losses in the denominator, as given by equation 4, is equal to $1 - \gamma$. Equation 4 together with the conditions on risk aversion limits the variability of risk aversion across states. Thus, the loss aversion coefficient γ takes values between zero and one. How much loss averse the agent is, depends on the sharpness of the kink in preferences, which is determined by the ratio of α to α_l. Consequently, the agent's degree of loss aversion γ is given by[9]:

$$\gamma = 1 - \frac{\alpha}{\alpha_l} \tag{4}$$

Where $\gamma \epsilon [0,1)$ represents the coefficient of loss aversion.

3.2 Equilibrium

The representative agent maximizes his utility U_t subject to his wealth accumulation constraint (equation 5).

$$\max_{c_t} U_t = \left(c_t^{\rho(s_t)} + \left(\beta(\mu_t)^{\rho(s_t)}\right)\right)^{\frac{1}{\rho(s_t)}} \tag{5}$$

$$s.t. \qquad x_{t+1} = (x_t - c_t) r_{t+1}$$

[9] *Please refer to Andries (2012) for more details.*

Where x_t is the beginning-of-period wealth, x_{t+1} is the representative agent's wealth at time $t+1$ and r_{t+1} is the gross real return of the stock market index.

Equity is the only asset in non-zero supply. Solving the agent's maximization problem yields the following Euler equations as first order conditions:

$$
\begin{aligned}
&E_t\left(\beta g_{t+1}^{\alpha_l-1}\left(\frac{\beta}{PD_t}\right)^{\frac{\alpha_l}{\rho(s_t)}-1}(1+PD_{t+1})^{\frac{\alpha_l}{\rho(s_{t+1})}-1}M_{t,t+1}\right)=1 && U_{t+1}<Ref_t\\
&E_t\left(\beta g_{t+1}^{\alpha_l-1}\left(\frac{\beta}{PD_t}\right)^{\frac{\alpha_l}{\rho(s_t)}-1}(1+PD_{t+1})^{\frac{\alpha_l}{\rho(s_{t+1})}-1}rf_{t+1}\right)=1 && U_{t+1}<Ref_t\\
&E_t\left(\beta^{\frac{\alpha_l}{\rho(s_t)}}g_{t+1}^{\alpha-1}(PD_t)^{\alpha_l-\alpha-\frac{\alpha_l}{\rho(s_t)}+1}(1+PD_{t+1})^{\frac{\alpha}{\rho(s_{t+1})}-1}Ref^{\alpha_l-\alpha}M_{t,t+1}\right)=1 && U_{t+1}\geq Ref_t\\
&E_t\left(\beta^{\frac{\alpha_l}{\rho(s_t)}}g_{t+1}^{\alpha-1}(PD_t)^{\alpha_l-\alpha-\frac{\alpha_l}{\rho(s_t)}+1}(1+PD_{t+1})^{\frac{\alpha}{\rho(s_{t+1})}-1}Ref^{\alpha_l-\alpha}rf_{t+1}\right)=1 && U_{t+1}\geq Ref_t
\end{aligned}
\tag{6}
$$

Where PD_t is the price dividend ratio at time t and g_{t+1} is the gross real consumption per capita growth rate at time $t+1$, $M_{t,t+1}=r_{t+1}$ is the gross return of the market index from time t to $t+1$ and rf_{t+1} is the gross return on the riskless asset.

Hence, under the same equilibrium conditions of Melino and Yang (2003), the Stochastic Discount Factor is:

$$
Q_{t,t+1}=
\begin{cases}
\beta g_{t+1}^{\alpha_l-1}\left(\frac{\beta}{PD_t}\right)^{\frac{\alpha_l}{\rho(s_t)}-1}(1+PD_{t+1})^{\frac{\alpha_l}{\rho(s_{t+1})}-1} & U_{t+1}<R\\
\beta^{\frac{\alpha_l}{\rho(s_t)}}g_{t+1}^{\alpha-1}(PD_t)^{\alpha_l-\alpha-\frac{\alpha_l}{\rho(s_t)}+1}(1+PD_{t+1})^{\frac{\alpha}{\rho(s_{t+1})}-1}Ref^{\alpha_l-\alpha} & U_{t+1}\geq R
\end{cases}
\tag{7}
$$

Proof: See Appendix C.

While the reference point at equilibrium will be:

$$
Ref_t = e^{E_t\left[\log\left(g_{t+1}*\frac{(1+PD_{t+1})^{\frac{1}{\rho(s_{t+1})}}}{PD_t}\right)\right]}
\tag{8}
$$

The utility function of the loss averse consumer implies a countercyclical risk aversion. It has been emphasized throughout the literature the need for countercyclical risk aversion to better match empirical data. In Melino-Yang state dependent preferences, the stochastic discount factor in the high state of nature is a function of the high state risk aversion value. Including loss aversion in the model allows the stochastic discount factor in the high state to be explicitly a function of high state as well as low state values of risk aversion. Hence, how much the agent is risk averse in the low state affects the high state stochastic discount factor.

4. Data and Methodology

This paper studies historical annual time series of the United States, the United Kingdom (UK), France, Germany, Italy, and Canada. Each of these countries alone represents more than one percentage of the capitalized global equity market. Together they constitute on average fifty eight percent of the capitalized global equity value throughout the period 1988-2012[10].

In addition, Italy is an example of a market where government's Treasury bill has on average a higher real risk-free return than the stock market index. I could not extend the sample to go further than 2012 because of my inaccessibility to these data. The motivation of examining the annual time series is that consumption for non-durables and services and population time series is available in annual frequency for all investigated countries.

The equities examined are market indices which represent the stock market. Standard and Poor's 500 composite price index; hereafter S&P 500, represents the United States stock market index. While, FTSE 100 index, DAX, CAC40, FTSE All-World Europe index[11] and S&P/TSX Composite act as a benchmark for the stock market index of UK, Germany, France, Italy, and Canada, respectively.

The sample is unbalanced. The period studied for each country depends on data availability. Data time series starts from the first year in which data on all variables are available. Data on the UK are available during the period 1988-2012, Germany during 1995-2012, France for the period 1990-2012, Italy over the period 1999-2012, Canada from 1988-2012 and the United States during the time span from 1890 to 2012.

The United States has the longest series of 123 years. Moreover, for the United States, I examine the subsample starting from 1890 until 1978 because it is the sample originally examined in Mehra and Prescott. Finally, I examine a common sample (1999-2012) to be able to compare results across countries.

Throughout the paper, the variables are in real values and in annual frequency. The variables used are as follows:

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[10]Source: World Bank database.

[11]The FTSE All-World Index Series is the European Large/Mid Cap stocks from the FTSE Global Equity Index Series. It covers 90-95% of the invesTable market capitalization universe for developed and emerging market segments.

(a) A real risk-free rate: it refers to the real yield on riskless short-term security. Each government's three-month Treasury bill represents riskless assets. But in the United States, prior to 1920, the proxy for the risk-free security was the two-months to three-month prime commercial paper. While for the period 1920-1930 Treasury Certificates represented riskless security. Cochrane and Hansen (1992) emphasized that the attempts to resolve the puzzle by accounting for mispricing of T-Bills are not likely to be productive.

(b) Real return of the stock market index: it serves as a proxy for the return of a risky asset representing the market. All indices' returns are adjusted for their dividends. I calculate S&P 500 and S&P/TSX Composite returns using dividends, while for other indices I use the total return index instead, because time series on dividends are missing. Whenever data on dividends is available, I calculate the stock market index gross returns as: $r_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$, where p_t is the stock market index's price at time t and d_{t+1} refers to the index's dividends at time $t+1$. Otherwise, I use the total return index for which gross returns on the market index equal to: $r_{t+1} = \frac{RI_{t+1}}{RI_t}$, where RI_t is the value of the total return index at time t.

(c) Real consumption per capita growth: it refers to the growth rate of real per capita consumption of non-durables and services based on each country's local currency.

(d) Real equity premium is the excess real return of the stock market index over the real risk-free rate in each country's local currency.

(e) Consumer Price index: these series are used directly to calculate the real values of the economic series. 2005 is the base year. In addition, its growth rate is used to calculate the inflation rate.

The United States data before 1969 is from the website of R. Mehra. Consumer price index data is from the World Bank database. On the other hand, data for the consumption per capita on non-durables and services for the United States is from the Bureau of Economic Analysis "BEA." For other variables, data source is DataStream.

4.1 Methodology

First, to define the threshold between low and high state concretely, I use a simple threshold model. Low and high states are defined with respect to the state variable. The threshold value does not affect the results. The next step is to calculate the low and high values for real consumption per capita growth and the real riskless assets. I calculate these two values for each variable mathematically matching its first two moments: namely its mean and variance, by solving the equations 9. Similarly, I calculate the price dividend ratio for each state, matching the first two moments of market returns.

A demonstration of the relation between market real returns and price dividend ratio at equilibrium is provided in appendix C. Market real returns can be easily recovered using equation A23. Let L and H represent the value of any variable "x" in the low and high state, then the system of equations to be solved (as given by equation 1 in Melino and Yang (2003)) is:

$$0.5\,L + 0.5\,H = E(x)$$
$$0.5\big(L - E(x)\big)^2 + 0.5\,\big(H - E(x)\big)^2 = V(x) \qquad (9)$$

where $E(x)$ refers to the variable's expected value, $V(x)$ represents its variance and 0.5 is the unconditional probability.

Finally, I calibrate the theoretical model. Substituting a constant value for the subjective discount factor β, I solve the model for the low and high values of risk aversion and elasticity of intertemporal substitution. In solving the system of Euler equations (equation 6) initial values are chosen carefully. I look for values of risk aversion and elasticity of intertemporal substitution which best satisfy the Euler equations. The loss aversion coefficient can be obtained from equation 3.4[12]. I keep the subjective discount rate fixed in comparability of results with the state-dependent preferences model.

5. Empirical Results

Melino and Yang (2003) demonstrated that the subjective discount factor should be $0.97 \leq \beta < 1$ for the elasticity of intertemporal substitution parameters to fall in the accepTable range, i.e., to have a value less than one. Hence, I set β equal to 0.98. As a robust check I derive the results for β equal to 0.99 as well. A subjective discount factor of 0.99 slightly affects the results but the norm of the Euler equations is equal to fixing its value to 0.98.

Using a simple threshold model, the low state realizes when real consumption per capita growth at time $t-1$ is less than a value of one for the United States, Canada as well as France and less than a value of zero for the rest of the sample. The threshold value does not affect the results. I calculate it to have a concrete definition of the difference between low and high state.

First, I examine the United States subsample 1890-1978 to compare results of the state dependent preferences model with and without loss aversion. Table 1 reports the low and high values of real consumption per capita growth, price-dividend ratio, and real risk-free rate as well as the real returns matrix.

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[12] *In Benartzi and Thaler (1995) with myopic loss aversion γ (using Andries definition) was estimated to be 0.36. On the other hand, Kahneman and Tversky (1979) estimated the loss aversion coefficient to be 0.44.*

Table 1. *Real consumption per capita growth, price-dividend ratio, real risk-free rate & real returns (The United States: 1890-1978)*

State of Nature	**Low**	**High**
Variables		
Real gross per capita consumption growth	0.98	1.054
Price-Dividend ratio	23.5	27.8
Real gross risk-free rate	0.95	1.064

Real Returns		
Next period	Low	High
Current		
Low	2.4%	30%
High	-0.1%	9%

Source: *Own study.*

Table 2 presents the calibration results. By construction, the representative agent for the United States (1890-1978) is risk averse in both states of nature. The difference in the agent's risk aversion across states is decimal as expected from the theoretical setting. Accordingly, the kink in preferences is not sharp. The elasticity of intertemporal substitution is almost constant in this subsample. Increasing the subjective discount factor decreases the elasticity of intertemporal substitution and increases the agent's risk aversion. The values of risk aversion are plausible according to the mainstream.

However, regardless of the value of β, the model produces a stochastic discount factor which barely varies across states. Hence, it violates Hansen-Jagannathan bound. From equation 4, the loss aversion coefficient γ is calculated to be 0.002 and 0.004 for β equal to 0.98 and 0.99, respectively. This value is lower than the empirical estimates in the literature which are around 0.55.

Table 2. *Model calibration results for the United States (1890-1978)*

	β=0.98		**β=0.99**	
	Low state	**High state**	**Low state**	**High state**
Risk Aversion	1.713	1.71	2.43	2.42
Elasticity of Intertemporal substitution	0.484	0.49	0.36	0.364

	β=0.98		**β=0.99**	
SDF(t+1)	**Low state**	**High state**	**Low state**	**High state**
SDF(t)				
Low state	1	0.89	1.03	0.88
High state	1.04	0.9	1.06	0.9

Source: *Own study.*

Loss aversion lowers the value of the agent's risk aversion compared to the Melino-Yang model. But the variation across states of risk aversion, elasticity of intertemporal substitution and the stochastic discount factor are higher in Melino-Yang model. I solved the model using Kahneman and Tversky lost adverse S-shaped preferences as well, but the model had no real solution in this case. Thus, my model with loss aversion lowers the representative agent's risk aversion as expected. This is good news for the loss aversion-state dependent recursive preferences model. But does the model match all asset market characteristics?

I extract from data the implied values of the agent's stochastic discount factor which satisfy the Euler equations disregarding the functional form of the stochastic discount factor. Table 3 shows that the implied stochastic discount factor is volatile. This provides further evidence for the conclusions of Hansen and Jagannathan (1991). As can be seen from Table 2, the model's stochastic discount factor modestly varies across states.

In addition, the model with loss aversion could not capture the negative value for the implied stochastic discount factor going from low to high state. This is because the stochastic discount factor of asset pricing models is positive by construction. Despite the good news of generating a low value for risk aversion, the model's stochastic discount factor values are far enough from their implied real values.

Table 3. *Implied stochastic discount factor values*

t+1 / t	Low state	High state
Low state	3.1	-0.5
High state	0.2	1.9

Source: *Own study.*

Second, to provide evidence if the results are country specific, I investigate the model for the United Kingdom, Germany, France, Canada, and Italy as well as the United States over the period 1890-2012. This sample is unbalanced so results are not for comparison across countries. I assume there is no trade, consequently I study the market of each country standing alone.

Tables in appendix D report the low and high values of all the key economic variables as well as the real returns matrix. It is worthy to mention, real risk-free rate has positive values in both high and low states for Italy and the United Kingdom, while both values are negative in France. In fact, France has a negative risk-free rate on average for the sample covered.

Real equity premium varies significantly across states. Despite the different time span, in all the investigated countries, except for Italy, real equity premium is only negative going from a high to a low state of nature. Contrarily, the Italian real equity premium is only positive going from a low to a high state. With a low current state and high next period's state, markets experience high real equity premium ranging between 47.9% in Germany and 33.3% in Italy. On the other hand, real equity premium is highly negative in all markets when current state is high and upcoming state is low fluctuating between -31.7% in Italy and -21.1% in France.

Table 4 presents the model's calibration results for risk aversion and elasticity of intertemporal substitution which best satisfy the Euler equations. Notably, a loss averse agent has a risk aversion value within accepTable limits in all countries. Subjective discount factor value affects the magnitude of risk aversion and the elasticity of intertemporal substitution. Its effect differs across countries. But still the difference between low and high values of risk aversion and the elasticity of intertemporal substitution is decimal.

Table 4. *Loss aversion model results*

	β=0.98		β=0.99	
	Low state	High state	Low state	High state
Risk Aversion				
United States (1890-2012)	1.01	1.01	2.14	2.13
United Kingdom (1988-2012)	0.57	0.58	0.27	0.27
Germany (1995-2012)	-0.31	-0.31	0.54	0.56
France (1990-2012)	3.9	3.8	4.3	4.29
Italy (1999-2012)	0.955	0.955	0.97	0.97
Canada (1988-2012)	3.2	3.2	2.9	2.9
Elasticity of Intertemporal Substitution				
United States (1890-2012)	0.97	0.97	0.23	0.26
United Kingdom (1988-2012)	0.056	0	0.068	0
Germany (1995-2012)	0.09	0.06	0.03	0.003
France (1990-2012)	0.054	0.07	0.04	0.06
Italy (1999-2012)	0.978	0.976	0.96	0.96
Canada (1988-2012)	0.055	0.07	0.046	0.06

Source: *Own study.*

Recovering γ from equation 4, the model suggests zero loss aversion in Canada and Italy for both values of subjective discount factor. While in the United Kingdom, loss aversion coefficient is 0.02 and 0 for β=0.98 and β=0.99 respectively. It takes a value of 0.03 and 0.002 in France, 0 and 0.005 in the United States and 0 and 0.04 in Germany for β=0.98 and β=0.99 respectively.

The model generates a stochastic discount factor which also differs decimally across states. All the stochastic discount factors are positive and close to one in all the countries investigated; its values vary in a range between 0.7 and 1.22 (Table 5). With this low level of variation, the model's stochastic discount factor values violate Hansen-Jagannathan bound.

Table 5. *Stochastic discount factor with loss aversion*

		β=0.98		β=0.99	
SDF(t+1) SDF(t)		**Low state**	**High state**	**Low state**	**High state**
United States	**Low**	0.98	0.9	0.996	0.89
	High	1.014	0.93	1.03	0.92
United Kingdom	**Low**	0.95	0.78	0.94	0.86
	High	1.11	0.91	1.04	0.95
Germany	**Low**	0.94	0.79	0.96	0.71
	High	1.14	0.96	1.22	0.89
France	**Low**	0.98	0.85	0.99	0.85
	High	1.13	0.98	1.13	0.97
Italy	**Low**	1	0.85	0.997	0.85
	High	1.1	0.93	1.1	0.93
Canada	**Low**	0.97	0.82	0.97	0.79
	High	1.1	0.93	1.1	0.92

Source: *Own study.*

I report on the implied stochastic discount factor values in Table 6. Comparing Table 5 with Table 6, it is noticeable that the model's stochastic discount factor values have a significantly lower variation than their implied counterparts. In addition, for the current high state of nature, the model generates countercyclical stochastic discount factor compared to the cyclical variation in the implied stochastic discount factor.

The model's stochastic discount factor is non-negative by construction. With loss aversion, the model could not even produce values close to zero whenever the stochastic discount factor should be negative.

This shows that, under the assumptions of my model, loss aversion lowers risk aversion but fails in matching other market characteristics. Using Kahneman and Tversky S-shaped preferences, the model has no real solution for all the countries investigated.

Table 6. *Stochastic discount factor implied values.*

SDF(t)	SDF(t+1)	**Low state**	**High state**
United States	**Low**	3.13	-0.64
	High	0.08	1.96
United Kingdom	**Low**	1.8	-0.4
	High	0.18	1.36
Germany	**Low**	2.1	-0.17
	High	0.19	1.7
France	**Low**	3	-0.3
	High	0.4	1.96
Italy	**Low**	2.6	0.096

	High	-0.2	3
Canada	Low	1.67	-0.48
	High	0.169	1.3

Source: *Own study.*

Third, I examine the common sample of all the investigated countries (1999-2012). In this sample, results can be compared across countries. This time is particular; it has witnessed the Lehman Brothers financial crisis (in 2008) as well as the European sovereign debt crisis. Table 7 presents the results of risk aversion and elasticity of intertemporal substitution in the common sample for all the investigated countries.

Table 7. *Common Sample (1999-2012): Risk Aversion and Elasticity of Intertemporal Substitution*

		β=0.98		β=0.99	
		Low state	High state	Low state	High state
Risk Aversion					
	United States	6.04	6.01	2.733	2.729
	United Kingdom	0.72	0.72	0.82	0.83
	Germany	2.128	2.127	3.764	3.763
	France	0.22	0.23	2.96	2.957
	Italy	0.955	0.955	0.97	0.97
	Canada	11.3	11.2	21.2	21.1
Elasticity of Intertemporal Substitution					
	United States	0.01	0.03	0.74	0.73
	United Kingdom	0.11	0	0.11	0
	Germany	0.19	0.2	0.5	0.48
	France	0.18	0.16	0.66	0.64
	Italy	0.978	0.976	0.96	0.96
	Canada	0.03	0.04	0.02	0.03

Source: *Own study.*

In the common sample, average risk aversion varies across countries. It ranges between 0.23 in France and 11.3 in Canada for a subjective discount factor of 0.98. While it ranges between 0.82 in the United Kingdom and 21.2 in Canada for $\beta = 0.99$. During the period 1999-2012, the representative agent of Canada has the highest average level of risk aversion which equals 11 for $\beta = 0.98$ and 21 when $\beta = 0.99$.

Risk aversion and elasticity of intertemporal substitution vary decimally across states. Italy has the highest elasticity of intertemporal substitution which is unit elastic. Elasticity of intertemporal substitution in France, Germany and the United States is sensitive to the value of the subjective discount factor. Risk aversion in the common sample is higher than its level in the longer sample examined in section 5.2 with France as the only exception.

6. Conclusion

This research paper examines the effect of adding loss aversion to the state-dependent preferences. The answer to the research question is No. Introducing loss aversion to state dependent recursive preferences lowers risk aversion level of the representative agent significantly even though the representative agent is assumed to be risk averse in both states. But the model fails in matching the implied stochastic discount factor values. Hence, in lowering the risk aversion, the model violates other characteristics of the market. The S-shaped preferences have no real solution and could not answer the research question either.

On one hand, introducing loss aversion, with a representative agent who is risk averse in both states, to state dependent recursive preferences lowers the agent's risk aversion significantly compared to the standard state dependent recursive preferences and the standard recursive preferences. Moreover, risk aversion is within what the mainstream considers as accepTable limits except for Canada in the common sample (1999-2012) with a 1% discount.

The model produces a loss aversion coefficient (Kahneman and Tversky's definition) close to one. In the literature this value is found to be around two. Notably, the exercise in this research paper is different from the literature as I extract its value from historical data under the assumptions of the proposed model.

On the other hand, the model generates a stochastic discount factor which varies modestly across states. Its values are not close enough to their implied counterparts and violates Hansen-Jagannathan bound. Besides, the variation in the agent's risk aversion across states is decimal. This limitation to the variability of risk aversion across states is expected from theoretical construction.

Increasing the subjective stochastic discount factor from 1% discount to 2% affects risk aversion and elasticity of intertemporal substitution levels. However, both values produce the same norm for the Euler equations. With loss aversion, the model's Euler equations have a norm of approximately 0.2 in all examined countries; except for Canada where the norm is 0.1.

This high value for the norm reflects the modest variation in the agent's risk aversion as well as stochastic discount factor unlike empirical data. Hence, these results are robust to the subjective stochastic discount factor value.

Therefore, a low risk aversion value is not the answer to the equity premium puzzle. Nevertheless, studying the effect of loss aversion on recursive preferences gives delightful insights and better understanding of many details to the problem in hand.

Conflict of interest: none.

References:

Andries, M. 2012. Consumption-Based Asset Pricing with Loss Aversion. Midwest Finance Association 2013 Annual Meeting paper.

Barberis, N., Ming, H. 2009. Preferences with Frames: A new Utility Specification that Allows for the Framing of Risks. Journal of Economic Dynamics and Control 33, 1555-1576.

Barberis, N., Ming, H., Tano, S. 2001. Prospect Theory and Asset Prices. The Quarterly Journal of Economics 116(1), 1-53.

Benartzi, S., Thaler, R. 1995. Myopic Loss Aversion and Equity Premium Puzzle. The Quarterly Journal of Economics 110(1), 73-92.

Campbell, J. 1999. Asset Prices, Consumption and Business Cycle. In: J.B. Taylor, M. Woodford (eds.), Handbook of Macroeconomics, Vol. 1C, Amsterdam: North Holland, pp. 1231-1303.

Cochrane, J., Lars, H. 1992. Asset Pricing Explorations for Macroeconomics. NBER Macroeconomics Annual, 7, 115-165.

Dimson, E., Marsh, P., Staunton, M. 2011. Equity Premia around the World. SSRN, October 7.

Dolmas, J. 2012. Campbell and Cochrane Meet Melino and Yang: Reverse Engineering the Surplus Ratio in a Mehra-Prescott Economy. Federal Reserve Bank of Dallas, Working Paper No. 1205.

Epstein, L., Stanley, Z. 1989. Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. Econometrica, 57(4), 937-969.

Epstein, L., Stanley Z. 1991. Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. The Journal of Political Economy, 99(2), 263-286.

Favero, C. 2001. Applied Macroeconometrics. Oxford, OX2: Oxford University Press.

Fielding, D., Stracca, L. 2007. Myopic Loss Aversion, Disappointment Aversion and the Equity Premium Puzzle. Journal of Economic Behavior & Organization, 64, 250-268.

Friend, I., Marshall, B. 1975. The Demand for Risky Assets. The American Economic Review, 65(5), 900-922.

Hansen, L., Ravi, J. 1991. Implications of Security Market Data for Models of Dynamic Economics. Journal of Political Economy, 99(2), 225-262.

Kahneman, D., Tversky, A. 1979. Prospect Theory: An Analysis of Decision under Risk. Econometrica, 47(2), 263-292.

Koszegi, B., Rabin, M. 2006. A Model of Reference-Dependent Preferences. Quarterly Journal of Economics, 121(4), 1133-1165.

Lucas, R. 1978. Asset Prices in an Exchange Economy. Econometrica, 46(6), 1429-1445.

Mehra, R. 2003. The Equity Premium: Why Is It a Puzzle? Financial Analysts Journal, 59(1), 54-69.

Mehra, R., Prescott, E. 1985. The Equity Premium: A puzzle. Journal of Monetary Economics, 15, 145-161.

Melino, A., Yang, A. 2003. State Dependent Preferences Can Explain the Equity Premium Puzzle. Review of Economic Dynamics, 6(4), 806-830.

Ross, S. 2015. The Recovery Theorem. The Journal of Finance, 70(2), 615-648.

Routledge, B., Stanley, Z. 2010. Generalized Disappointment Aversion and Asset Prices. The Journal of Finance, 65(4), 1303-1332.

Thaler, R., Johnson, E. 1990. Gambling with House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice. Management Science 36(6), 643-660.
Tversky, A., Kahneman, D. 1992. Advances in Prospect Theory: Cumulative Representation of Uncertainty. Journal of Risk and Uncertainty, 5, 297-323.
Weil, P. 1990. Non-expected Utility in Macroeconomics. The Quarterly Journal of Economics, 105(1), 29-42.

Appendix A:

Given preferences as follows:

$$U_t(s_t) = \left(c_t^{\rho(s_t)}(s_t) + (\beta {\mu_t}^{\rho(s_t)}(s_t))\right)^{\frac{1}{\rho(s_t)}}$$

Where μ_t is the certainty equivalence function at time t. $c_t$is current consumption.

Then

$$U_{t+1}(s_{t+1}) = f(\, c_{t+1}(s_{t+1}), \mu_{t+1}(s_{t+1})) = \left(c_{t+1}^{\rho(s_{t+1})} + (\beta {\mu_{t+1}}^{\rho(s_{t+1})})\right)^{\frac{1}{\rho(s_{t+1})}}$$

In the low state of nature both c_{t+1} and μ_{t+1} are expected to be lower than in the high state. A superscript l refers to the value of the variable in the low state of nature while a superscript h refers to its value in the high state of nature.

$\because\ c_{t+1}^l < c_{t+1}^h$
$\mu_{t+1}^l < \mu_{t+1}^h$

And given more consumption is preferred to less,

$$\therefore\ U_{t+1}^l < U_{t+1}^h \qquad \text{(A1)}$$

Given the reference point as:

$$Ref_t = e^{E_t[\log(U_{t+1}(s_{t+1}))]}$$

For the representative agent to be loss averse in the current low state of nature, it requires U_{t+1}^l to be lower than Ref_t.

$$U_{t+1}^l < e^{E_t[\log(U_{t+1}(s_{t+1}))]}$$

Suppose not.
So, suppose $U_{t+1}^l > e^{E_t[\log(U_{t+1}(s_{t+1}))]}$

By taking logs we get

$\log(U_{t+1}^{l}) > E_t[\log(U_{t+1}(s_{t+1}))]$

But this cannot be true because $E_t[\log(U_{t+1}(s_{t+1}))]$ is a convex combination of $\log(U_{t+1}^{l})$ and $\log(U_{t+1}^{h})$ and from (A1) we know that $\log(U_{t+1}^{l})$ has a lower value than $\log(U_{t+1}^{h})$.

Q.E.D.

Appendix B:

In deriving the certainty equivalent function, as shown in Andries (2012), I mention first an example for the standard case then the multi-period recursive case.

The standard CRRA preferences with loss aversion are given by the following function:

$$U(c, R(c)) = \begin{cases} \frac{c^{\alpha_l}}{\alpha_l} & for\ c \leq R(c) \\ \frac{c^{\alpha}}{\alpha} & for\ c > R(c) \end{cases}$$

Where c is consumption and $R(c)$ is the reference point.

At the reference point, we get:

$$\frac{\alpha_l}{\alpha} = R(c)^{\alpha_l - \alpha}$$

Hence, we can rewrite the preferences as:

$$U(c, R(c)) = \frac{1}{\alpha_l}\begin{cases} c^{\alpha_l} & for\ c < R(c) \\ c^{\alpha} * R(c)^{\alpha_l - \alpha} & for\ c \geq R(c) \end{cases} \quad \text{(A2)}$$

For the multi-period recursive preferences:

$$U_t = \left(c_t^{\rho(s_t)} + (\beta \mu_t{}^{\rho(s_t)})\right)^{\frac{1}{\rho(s_t)}}$$

Where μ_t is the CRRA certainty equivalence function. With loss aversion, the specification of U_{t+1} will differ in the following way:

$$\begin{cases} U_{t+1}^{\alpha_l} & for\ U_{t+1} < Ref_t \\ U_{t+1}^{\alpha} & for\ U_{t+1} \geq Ref_t \end{cases}$$

Where Ref_t is the reference point.
Analogously to the standard case, the certainty equivalence function can be rewritten in the following way:

$$\mu_t = \begin{cases} \left(E_t(U_{t+1}^{\alpha_l})\right)^{\frac{1}{\alpha_l}} & U_{t+1} < Ref_t \\ \left(E_t(U_{t+1}^{\alpha} * Ref_t^{\alpha_l - \alpha})\right)^{\frac{1}{\alpha_l}} & U_{t+1} \geq Ref_t \end{cases} \quad \text{(A3)}$$

Appendix C:

State dependent preferences with loss aversion first order conditions:

I used the proof provided in Melino and Yang (2003) for state dependent preferences first order conditions and applied it to the preferences after introducing loss aversion. For more details, please refer to Melino and yang (2003) and Epstein and Zin (1989).

I show the proof for the case with more than one risky asset, and it will surely apply for the case of just one risky asset.

The representative agent's objective is to maximize his utility while choosing his consumption level and portfolio shares. The bellman equation for this equation is:

$$J(x_t, I_t) = \max_{c_t, w_t} \left(c_t^{\rho(s_t)} + \beta \mu_t\left(J(x_{t+1}, I_{t+1})\right)^{\rho(s_t)}\right)^{\frac{1}{\rho(s_t)}} \quad \text{(A4)}$$

$$such\ that \quad x_{t+1} = (x_t - c_t) w_t r_{t+1} \quad \text{(A5)}$$

Where I_t denotes the information variable, w_t refers to the portfolio share vector, r_{t+1} is the gross real returns for risky assets and $\mu_t(.)$ is the certainty equivalence function where $\mu_t(\tilde{z})$ is given by eq. A6 for all random variables $\tilde{z}$. The bellman equation is function of x_t because it describes the state of the system at the beginning of t when the representative agent makes the decision. While c_t and w_t act as the control variables of the bellman equation because it is under the choice of the representative agent.

$$\mu_t(\tilde{z}) = \begin{cases} \left(E_t(\tilde{z}^{\alpha_l})\right)^{\frac{1}{\alpha_l}} & \tilde{z} < Ref_t \\ \left(E_t(\tilde{z}^{\alpha} * Ref_t^{\alpha_l - \alpha})\right)^{\frac{1}{\alpha_l}} & \tilde{z} \geq Ref_t \end{cases} \quad \text{(A6)}$$

Ref_t is the reference point.

The homogeneity of $\mu_t(.)$ and the linearity of x_{t+1} in (x_t, c_t) in eq. A5 implies the solution has the form:

$$J(x_t, I_t) = A(I_t)\, x_t \quad \text{(A7)}$$

Where $A(I_t)$ is an I_t-measurable variable.

Substituting eq. (A6) into eq. (A4) gives:

$$A(I_t)\, x_t = \max_{c_t, w_t} \left(c_t^{\rho(s_t)} + \beta\mu_t\big(J(x_{t+1}, I_{t+1})\big)^{\rho(s_t)}\right)^{\frac{1}{\rho(s_t)}} \quad \text{(A8)}$$

This maximization problem can be decomposed into two problems: consumption and the portfolio choice problem.

Consumption is chosen by:

$$A(I_t)\, x_t = \max_{c_t \in [0, x_t]} \left(c_t^{\rho(s_t)} + \beta[(x_t - c_t)\mu_t^*]^{\rho(s_t)}\right)^{\frac{1}{\rho(s_t)}} \quad \text{(A9)}$$

While portfolio choice is described by:

$$\mu_t^* = \begin{cases} \max_{w \in R_+^n, w\iota = 1} E_t[(A(I_{t+1}) w r_{t+1})^{\alpha_l}]^{\frac{1}{\alpha_l}} & \tilde{z} < Ref_t \\ \max_{w \in R_+^n, w\iota = 1} E_t[(A(I_{t+1}) w r_{t+1})^{\alpha} * Ref_t^{\alpha_l - \alpha}]^{\frac{1}{\alpha_l}} & \tilde{z} \geq Ref_t \end{cases} \quad \text{(A10)}$$

Where ι = (1, 1, …, 1)’ and $\tilde{z} = A(I_{t+1}) w r_{t+1}$.

First, let us solve the consumption problem for an arbitrary μ_t^*. The homogeneity of eq. A9 implies the optimal consumption policy can be written as:

$$c_t^* = a(I_t) x_t \quad \text{(A11)}$$

Where $a(I_t)$ is an I_t-measurable variable which can be thought of as the consumption wealth ratio (hereafter a_t).

Substituting eq. A11 into eq. A9:

$$A(I_t)^{\rho(s_t)} = \max_{a_t \in [0,1]} a_t^{\rho(s_t)} + \beta[(1 - a_t)\mu_t^*]^{\rho(s_t)} \quad \text{(A12)}$$

Then the F.O.C is:

$$a_t^{\rho(s_t)-1} = \beta(1 - a_t)^{\rho(s_t)-1} \mu_t^{*\rho(s_t)} \quad \text{(A13)}$$

Combining the last two equations yields:

$$A(I_t) = a_t^{\frac{\rho(s_t)-1}{\rho(s_t)}} \quad \text{(A14)}$$

Therefore,

$$A(I_{t+1}) = a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} \qquad \text{(A15)}$$

Now substitute eq. A15 into eq. A10 to get:

$$\mu_t^* = \begin{cases} \max\limits_{w\in R_+^n, w\iota=1} E_t\left[(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} wr_{t+1})^{\alpha_l}\right]^{\frac{1}{\alpha_l}} & \tilde{z} < Ref_t \\ \max\limits_{w\in R_+^n, w\iota=1} E_t\left[(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} wr_{t+1})^{\alpha} * Ref_t^{\alpha_l-\alpha}\right]^{\frac{1}{\alpha_l}} & \tilde{z} \geq Ref_t \end{cases} \qquad \text{(A16)}$$

Where $\iota = (1, 1, \ldots, 1)'$ and $\tilde{z} = a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} wr_{t+1}$.

The first sub-function of the piecewise certainty equivalence function, i.e., when $\tilde{z} < Ref_t$, is the same as the certainty equivalence function provided in Melino and Yang (2003). Substituting eq. A16 into eq. A13 gives:

$$a_t^{\rho(s_t)-1} = \begin{cases} \beta(1-a_t)^{\rho(s_t)-1} E_t\left[(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} M_{t,t+1})^{\alpha_l}\right]^{\frac{\rho(s_t)}{\alpha_l}} & \tilde{z} < Ref_t \\ \beta(1-a_t)^{\rho(s_t)-1} E_t\left[(a_{t+1}^{\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}} M_{t,t+1})^{\alpha} * Ref_t^{\alpha_l-\alpha}\right]^{\frac{\rho(s_t)}{\alpha_l}} & \tilde{z} \geq Ref_t \end{cases} \qquad \text{(A17)}$$

Where $M_{t,t+1} = w_t^* r_{t+1} = \sum_{i=1}^{n} w_{it}^* r_{i,t+1}$ is the gross return of the optimal portfolio w_t^* or in other words the market portfolio. In the case of one risky asset, which is the case studied in this paper, $M_{t,t+1} = r_{t+1}$.

Rearranging eq. A17, with just algebra, yields:

$$\begin{cases} E_t\left[\beta^{\frac{\alpha_l}{\rho(s_t)}} \left(\frac{1-a_t}{a_t}\right)^{\alpha_l(\frac{\rho(s_t)-1}{\rho(s_t)})} a_{t+1}^{\alpha_l(\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})})} M_{t,t+1}{}^{\alpha_l}\right] = 1 & \tilde{z} < Ref_t \\ E_t\left[\beta^{\frac{\alpha_l}{\rho(s_t)}} \left(\frac{1-a_t}{a_t}\right)^{\alpha_l(\frac{\rho(s_t)-1}{\rho(s_t)})} a_{t+1}^{\alpha(\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})})} M_{t,t+1}{}^{\alpha}\, Ref_t{}^{\alpha_l-\alpha}\right] = 1 & \tilde{z} \geq Ref_t \end{cases} \qquad \text{(A18)}$$

Notice that eq. A18 can be written in the form:

$$E_t[Q_{t,t+1} * M_{t,t+1}] = 1 \qquad \text{(A19)}$$

Where $Q_{t,t+1}$ (the stochastic discount factor) is:

$$Q_{t,t+1} = \begin{cases} \beta^{\frac{\alpha_l}{\rho(s_t)}} \left(\frac{1-a_t}{a_t}\right)^{\alpha_l\left(\frac{\rho(s_t)-1}{\rho(s_t)}\right)} a_{t+1}^{\alpha_l\left(\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}\right)} M_{t,t+1}{}^{\alpha_l - 1} & \tilde{z} < Ref_t \\ \beta^{\frac{\alpha_l}{\rho(s_t)}} \left(\frac{1-a_t}{a_t}\right)^{\alpha_l\left(\frac{\rho(s_t)-1}{\rho(s_t)}\right)} a_{t+1}^{\alpha\left(\frac{\rho(s_{t+1})-1}{\rho(s_{t+1})}\right)} M_{t,t+1}{}^{\alpha - 1}\, Ref_t{}^{\alpha_l - \alpha} & \tilde{z} \geq Ref_t \end{cases} \tag{A20}$$

The budget constraint (eq. A5) implies:

$$x_{t+1} = (x_t - c_t)M_{t,t+1} = (1 - a_t)M_{t,t+1}x_t$$

Therefore,

$$a_{t+1} = {}^{c_{t+1}}/_{x_{t+1}} = {}^{g_{t+1}a_t}/_{[(1 - a_t)M_{t,t+1}]} \tag{A21}$$

At equilibrium $c_{t+1} = d_{t+1}$ so the market portfolio is

$$M_{t,t+1} = \frac{p_{t+1} + c_{t+1}}{p_t} \tag{A22}$$

Where p_t is the price of equity at time t and d_{t+1} is the equity dividend at time $t + 1$. Following Melino and Yang (2003), equilibria are considered where ex-dividend price of equity $p(g_t, c_t)$ is described by the time invariant and positive function of the variables g_t and c_t. Due to homogeneity of preferences, the price is linearly homogeneous in consumption. So $p(g, c) = p\left(\frac{g}{c}, 1\right) * c = P(g) * c$ where $P(g)$ is the price-dividend ratio.

Hence the market portfolio can be written as:

$$M_{t,t+1} = \frac{P(g_{t+1}) + 1}{P(g_t)}\, g_{t+1} \tag{A23}$$

In Lucas endowment equilibrium, the agent's wealth is:

$$x_t = p_t + c_t$$

Therefore, the consumption-wealth ratio at equilibrium can be written as:

$$a_t \equiv \frac{c_t}{x_t} = \frac{1}{1 + {}^{p_t}/_{c_t}} = \frac{1}{1 + P_t} \tag{A24}$$

Substituting eq. A23 and A24 in eq. A20, $Q_{t,t+1}$ at equilibrium can be equivalent as:

$$Q_{t,t+1} = \begin{cases} \beta g_{t+1}^{\alpha_l - 1} \left(\frac{\beta}{PD_t}\right)^{\frac{\alpha_l}{\rho(s_t)}-1} (1+PD_{t+1})^{\frac{\alpha_l}{\rho(s_{t+1})}-1} & U_{t+1} < Ref_t \\ \beta^{\frac{\alpha_l}{\rho(s_t)}} g_{t+1}^{\alpha-1} (PD_t)^{\alpha_l - \alpha - \frac{\alpha_l}{\rho(s_t)}+1} (1+PD_{t+1})^{\frac{\alpha}{\rho(s_{t+1})}-1} Ref^{\alpha_l-\alpha} & U_{t+1} \geq Ref_t \end{cases} \quad \text{(A25)}$$

Appendix D:

Table D1. *Economic variables' values in each state of nature (The United States: 1890-2012)*

State of Nature	Low	High
Variables		
Real gross per capita consumption growth	0.99	1.048
Price-Dividend ratio	19.8	23.8
Real gross risk-free rate	0.94	1.082

Real Returns		
Next period	Low	High
Current		
Low	3.5%	31.7%
High	-0.14%	9%

Table D2. *Economic variables' values in each state of nature (The United Kingdom: 1988-2012)*

State of Nature	Low	High
Variables		
Real gross per capita consumption growth	0.99	1.031
Price-Dividend ratio	17.4	22.4
Real gross risk-free rate	1.001	1.06

Real Returns		
Next period	Low	High
Current		
Low	5.13%	38.9%
High	-0.19%	7.7%

Table D3. *Economic variables' values in each state of nature (Germany: 1995-2012)*

State of Nature	Low	High
Variables		
Real gross per capita consumption growth	0.98	1.019
Price-Dividend ratio	19.2	26.8
Real gross risk-free rate	0.997	1.027

Real Returns		
Next period	Low	High

Current		
Low	3.3%	47.5%
High	-0.26%	5.7%

Table D4. *Economic variables' values in each state of nature (France: 1990-2012)*

State of Nature	**Low**	**High**
Variables		
Real gross per capita consumption growth	0.99	1.019
Price-Dividend ratio	21.9	29
Real gross risk-free rate	0.98	0.99

Real Returns		
Next period	Low	High
Current		
Low	3.9%	39.8%
High	-0.22%	5.4%

Table D5. *Economic variables' values in each state of nature (Italy: 1999-2012)*

State of Nature	**Low**	**High**
Variables		
Real gross per capita consumption growth	0.97	1.017
Price-Dividend ratio	47	61.4
Real gross risk-free rate	1.016	1.077

Real Returns		
Next period	Low	High
Current		
Low	-0.01%	34.9%
High	-0.24%	3.3%

Table D6. *Economic variables' values in each state of nature (Canada: 1988-2012)*

State of Nature	**Low**	**High**
Variables		
Real gross per capita consumption growth	0.99	1.019
Price-Dividend ratio	16.4	21.6
Real gross risk-free rate	0.999	1.051

Real Returns		
Next period	Low	High
Current		
Low	5.2%	40.3%
High	-0.2%	6.6%

Appendix E:

The standard Epstein-Zin preferences are represented by the utility function as in equation A26.

$$U_t(c_t, \mu_t) = \left(c_t^{\rho} + (\beta(\mu_t)^{\rho})\right)^{\frac{1}{\rho}} \qquad \text{(A26)}$$

The elasticity of intertemporal substitution (EIS) is the percentage change in consumption growth relative to the percentage change in interest rate $EIS = \partial \, ln \, (c_{t+1}/c_t)/\partial \, ln \, R$. For deterministic consumption programs:

$$R = \frac{U'(c_t)}{\beta U'(c_{t+1})} \quad \text{(A27)}$$

$$\therefore \ln R = -\ln\left(\frac{U'(c_{t+1})}{U'(c_t)}\right) - \ln \beta \quad \text{(A28)}$$

The subjective discount factor is assumed to be constant.

$$\therefore EIS = -\partial \, ln \, (c_{t+1}/c_t)/\partial \, ln \, (U'(c_{t+1})/U'(c_t)) \quad \text{(A29)}$$

From eq. A26

$$ln \, (U'(c_{t+1})/U'(c_t)) = (\rho - 1) \, ln \, (c_{t+1}/c_t)) \quad \text{(A30)}$$

Substituting eq. A30 in eq. A29:

$$EIS = \frac{1}{1-\rho} \, \partial \, ln \, (c_{t+1}/c_t)/\partial \, ln \, (c_{t+1}/c_t) = \frac{1}{1-\rho} \quad \text{(A31)}$$

Q.E.D.